Rational Maps with Cluster Cycles and the Mating of Polynomials

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Dynamical Systems Seminar

Stony Brook University

Outline

1. Introduction
   - Standard Definitions
   - Matings
   - Thurston Equivalence
2. Clustering
   - Combinatorial data
3. Results
   - Thurston Equivalence
   - Fixed Cluster points
   - Period 2 cluster cycle results
   - Higher Degrees

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Definitions.

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map.

- The Julia set $J(f)$ is the closure of the set of repelling periodic points of f.
- The Fatou set $F(f)$ is $\widehat{\mathbb{C}} \setminus J(f)$.

If f is a polynomial

- The point ∞ is a superattracting fixed point.
- The filled Julia set is $K(f) = \{z \in \widehat{\mathbb{C}} \mid f^{\circ n}(z) \nrightarrow \infty\}$, so that $J(f) = \partial K(f)$

In this talk, we will generally assume that f has a (finite) superattracting periodic cycle of period $p > 1$.

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Böttcher's theorem and external rays

There exists an analytic conjugacy ϕ between f on $\widehat{\mathbb{C}} \setminus K(f)$ and the map $z \mapsto z^d$ on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

- ϕ is asymptotic to the identity at ∞.
- The set $R_\theta = \phi^{-1}\{re^{2\pi i\theta} \mid r \in (1, \infty)\}$ is called the external ray of angle θ.
- If $J(f)$ is locally connected, the point $\gamma(\theta) = \lim_{r\to 1} \phi^{-1}(re^{2\pi i\theta})$ exists for all θ and belongs to $J(f)$.
- We have the identities $f(R_\theta) = R_{d\theta}$ and $f(\gamma(\theta)) = \gamma(d\theta)$.
  - The points $\beta_k = \gamma(k/(d-1))$, $k = 0, 1, \ldots, d-2$ are fixed points on $J(f)$.
  - If $\alpha \in J(f)$ is another fixed point and $\alpha = \gamma(\theta)$, then $\alpha = \gamma(d\theta), \gamma(d^2\theta), \ldots$

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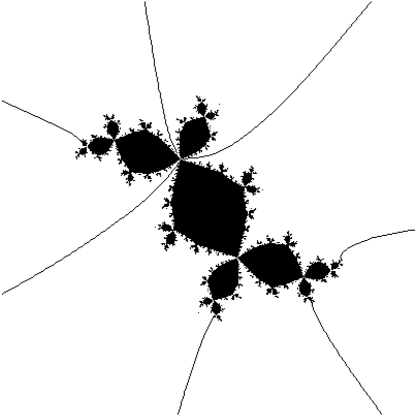

Douady's rabbit with the external rays of angle $p/7$.

Formal Matings

New maps from old

Denote $f_i(z) = z^d + c_i$. Define $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2\pi i s} \mid s \in \mathbb{R}/\mathbb{Z}\}$.

- Extend f_1 and f_2 to the boundary circle at infinity, e.g. $f_1(\infty \cdot e^{2\pi i s}) = \infty \cdot e^{2d\pi i s}$.
- Define $S^2_{f_1,f_2} = \widetilde{\mathbb{C}}_{f_1} \uplus \widetilde{\mathbb{C}}_{f_2} / \{(\infty \cdot e^{2\pi i s}, f_1) \sim (\infty \cdot e^{-2\pi i s}, f_2)\}$.
- The formal mating is a branched covering $f_1 \uplus f_2 \colon S^2_{f_1,f_2} \to S^2_{f_1,f_2}$.
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Topological Matings

Denote $f_i(z) = z^d + c_i$ with filled Julia set $K_i = K(f_i)$.

- Define a new map $f_1 \perp\!\!\!\perp f_2$ on $K_1 \perp\!\!\!\perp K_2$.
  - Take the disjoint union of K_1 and K_2.
  - $K_1 \perp\!\!\!\perp K_2$ is the quotient space formed by identifying $\gamma_1(\theta)$ with $\gamma_2(-\theta)$.
  - The maps f_i on the K_i fit together to form a new map $f_1 \perp\!\!\!\perp f_2$.

We say f_1 and f_2 are mateable if this quotient $K_1 \perp\!\!\!\perp K_2$ is a sphere.

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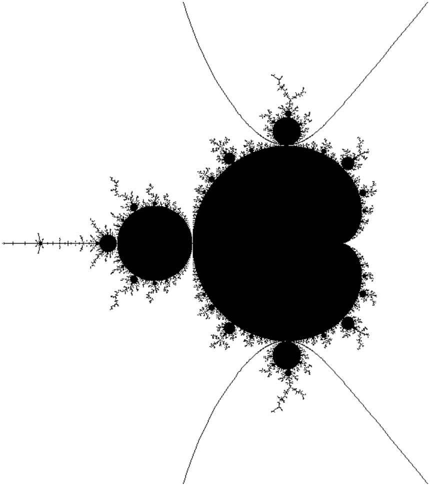
Obstructed matings

Let f_1 be the rabbit polynomial and f_2 the anti-rabbit polynomial.



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Let f_1 be the rabbit polynomial and f_2 the anti-rabbit polynomial.

- The quotient space created by the ray equivalence class is not homeomorphic to the sphere.
- So $f_1 \perp\!\!\!\perp f_2$ is not equivalent to a rational map.
- The same holds for any mating of f_1 with a map in the anti-rabbit limb.

Definition

A multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of F is called a Levy cycle if for $i = 1, 2, \ldots, n$, the curve γ_{i-1} is homotopic (rel P_F) to a component γ'_{i-1} of $F^{-1}(\gamma_i)$ and the map $F\colon \gamma'_i \to \gamma_i$ is a homeomorphism.

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Thurston's Theorem

Let $F\colon \Sigma \to \Sigma$ and $\widehat{F}\colon \widehat{\Sigma} \to \widehat{\Sigma}$ be postcritically finite orientation-preserving branched self-coverings of topological 2-spheres. An equivalence is given by a pair of orientation-preserving homeomorphisms (Φ, Ψ) from Σ to $\widehat{\Sigma}$ such that

- $\Phi|_{P_F} = \Psi|_{P_F}$
- $\Phi \circ F = \Psi \circ \widehat{F}$
- Φ and Ψ are isotopic via a family of homeomorphisms $t \mapsto \Phi_t$ which is constant on P_F.

Theorem (Thurston)

Let $F\colon \Sigma \to \Sigma$ be a postcritically finite branched cover with hyperbolic orbifold. Then F is equivalent to a rational map if and only if F has no Thurston obstructions. This rational map is unique up to Möbius transformation. In particular, any equivalence between two rational maps is realized by a Möbius transformation.

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Simplifying Thurston's criterion

In general it is difficult to find Thurston obstructions. Levy cycles simplify the search.

- F has a Levy cycle $\Rightarrow$ F has a Thurston obstruction.
- In the bicritical case: F has a Thurston obstruction $\Rightarrow$ F has a Levy cycle.

Theorem (Rees, Shishikura, Tan)

In the bicritical case, if $[\alpha_1] \neq [\alpha_2]$, $K_1 \amalg K_2$ is homeomorphic to S^2 and we can give this sphere a unique conformal structure to make $f_1 \amalg f_2$ a holomorphic degree d rational map.

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Clusters (Definition)

Let F be a bicritical rational map such that the two critical points belong to the attracting basins of two disjoint (super)attracting periodic orbits of the same period.

Clustering is the condition that the critical orbit Fatou components group together to form a periodic cycle...

- The dynamics on each Fatou component can be conjugated using Böttcher's theorem.
  - Internal rays
- The 0 internal ray is fixed under the first return map.
- If the 0 internal rays meet at a point c, and this point is periodic, we say c is a cluster point for F.

e.g. Rabbit ⫫ Airplane.

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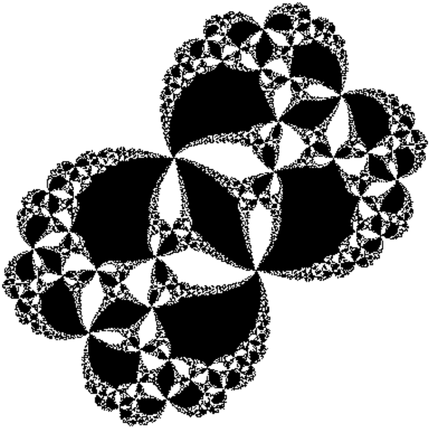
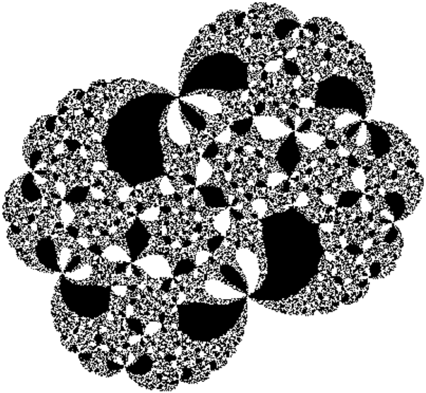
Example



The Julia set for Rabbit $\perp\!\!\!\perp$ Airplane
(and Airplane $\perp\!\!\!\perp$ Rabbit!).

Another example



The Julia set for a map with a period two cluster cycle.

Combinatorial data

Restrict attention to the period one and two cases.
We can describe a cluster in simple combinatorial terms.

1. (The period of the critical cycles n.)
2. The combinatorial rotation number ρ.
3. The critical displacement δ.

The combinatorial data of the cluster will be the pair (ρ, δ).

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Clusters and Thurston Equivalence

Theorem (S., 2010)

If F and G are bicritical rational maps of the same degree and

- *F and G have fixed cluster points and are of degree d*
- *F and G are quadratic with a period 2 cluster cycle*

then F and G have the same combinatorial data if and only if they are Thurston equivalent.

In particular, this means that a rational map is entirely determined by its behavior inside the cluster cycle.

The theorem is not true in the case where F and G have degree $d \geq 3$ and a period two cluster cycle (An example is given in joint work with Adam Epstein. . . more on this later.)

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Proof Outline

In both cases we use the following method. let X_F and X_G be the union of the stars of the clusters of F and G.

1. There exists $\phi\colon X_F \to X_G$ which conjugates the dynamics.
2. Extend ϕ to a map $\Phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.
3. Construct a new map $\widehat{\Phi}$ such that $\Phi \circ F = G \circ \widehat{\Phi}$.
4. Modify $\widehat{\Phi}$ so that it is equal to ϕ on X_F.
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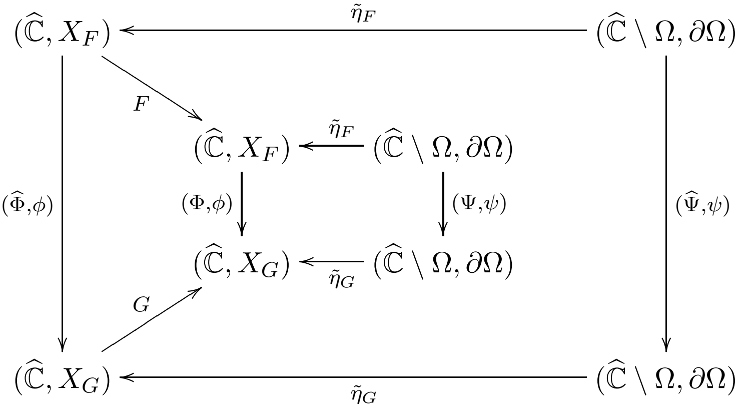
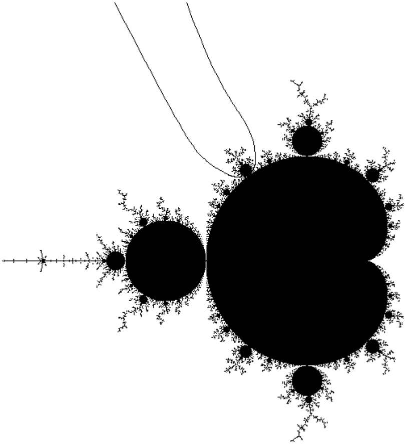
A bit more detail...



Ω is either $\mathbb{D}$ (fixed case) or an annulus (period 2 case).

Period 1 Results

A rabbit is any map with a “star-shaped” Hubbard tree. They belong to hyperbolic components which bifurcate from the (unique) period one component in the Multibrot set.



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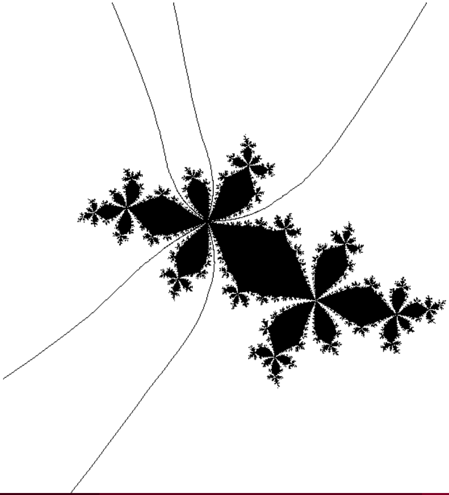

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Lemma

If $F = f_1 \amalg f_2$ has a fixed cluster point, then precisely one of the f_i is a rabbit.

Lemma

All combinatorial data can be realized (in precisely $2d-2$ ways), save for the case with $\delta = 1$ or $\delta = 2n-1$.

The rotation number is fixed by the rotation number of the α-fixed point for the rabbit. The critical displacement is determined by the choice of the complementary map.

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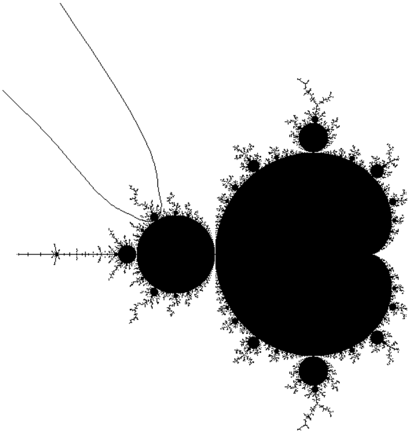
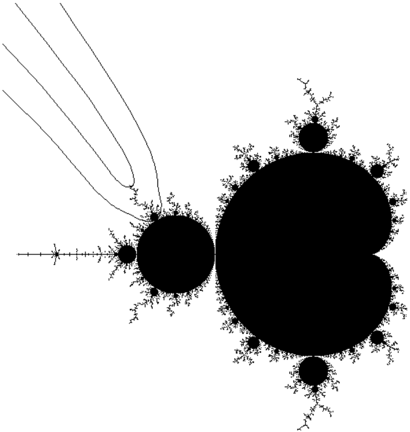
Period 2 (quadratic) Results

A bi-rabbit is a map bifurcating off the period 2 component. . .



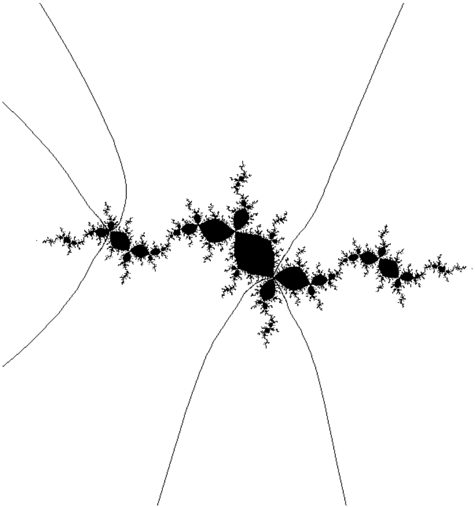
Period 2 (quadratic) Results

...but there is another component of the same period in the wake of the bi-rabbit!



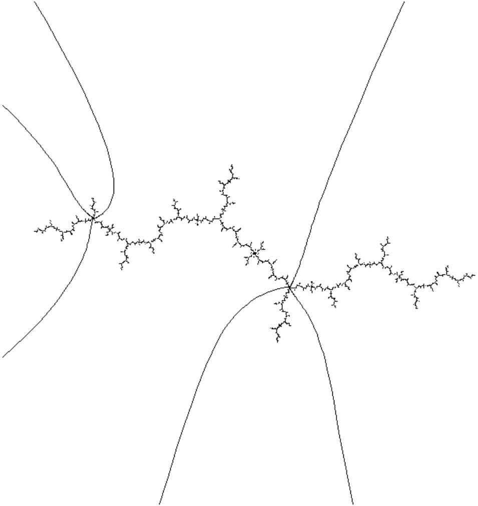
Period 2 (quadratic) Results

The rays landing on the period 2 cycle of the bi-rabbit. . .



Period 2 (quadratic) Results

...have the same angles as those landing on the period 2 cycle of the secondary map.



Period 2 (quadratic) Results

A bi-rabbit is a map bifurcating off the period 2 component.

Theorem (S., 2009)

If $F = f_1 \perp\!\!\!\perp f_2$ has a period two cluster cycle, one of the f_i is either a bi-rabbit or a secondary map which lies in the limb of the bi-rabbit.

Lemma (S., 2009)

All combinatorial data can be realized (in at least two ways).

Theorem (S., 2010)

The cases $\delta = 1$ and $\delta = 2n - 1$ can be constructed from mating with the secondary map.

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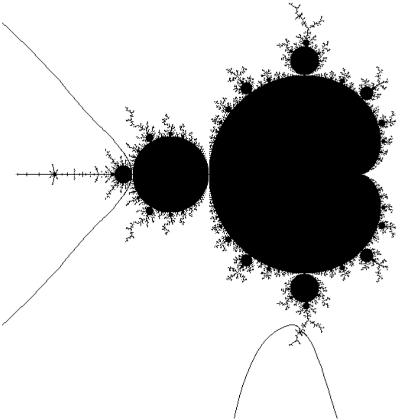
Example

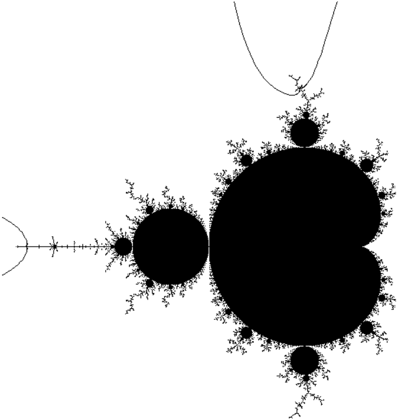
The mating of these two components. . .



This is a well-known example. . .

Example

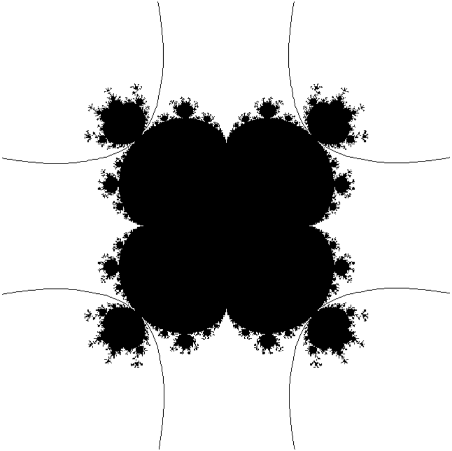
. . . is equivalent to the mating of these two components



This is a well-known example. . .

Higher degrees

Consider the degree d Multibrot set $\mathcal{M}_d$. It has $d-1$ period two hyperbolic components.



Joint work with Epstein

We can mate the corresponding period two polynomials (if they are not complex conjugate) to get a degree d rational map with a period two cluster cycle. Each cluster contains precisely one periodic Fatou component from each critical orbit.

- The rotation number must be $0/1$
- The displacement must be 1

But not all maps arising from these matings are equivalent! We can even write the maps in normal form as

$$F_a(z) = \frac{z^d - a}{z^d - 1},$$

where $a \neq 1$ is a $(d-1)$th root of unity.

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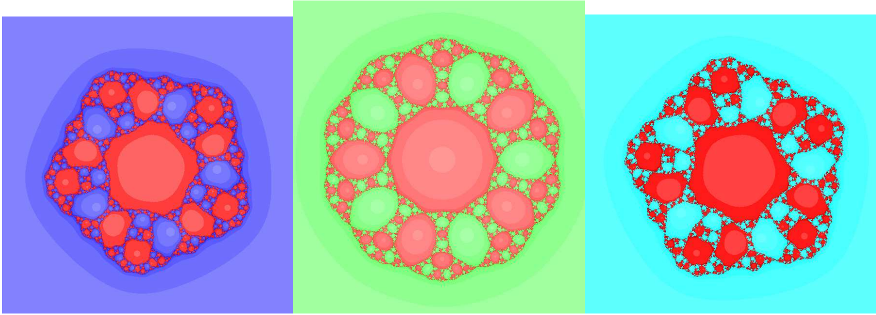
Joint work with Epstein

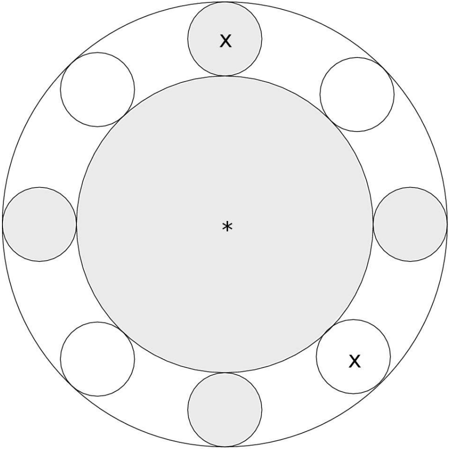
Here are the period 5 examples...



In general there are $d-2$ configurations in degree d.

A schematic proof

We can describe the Julia sets of the maps schematically



We have a modified notion of displacement. . . and this helps us classify the maps.

Summary

- In simple cases, the combinatorial data of a cluster completely defines a rational map.
  - Period 3 - experimental pictures suggest not?
- Period 1 and period 2 cases are very similar, but with an increased level of complexity for the period 2 case.
  - “Non-trivial” shared matings
  - Different combinatorial data
  - Simple Thurston classification only works in the quadratic case for period 2
- Combinatorics of the matings

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Outlook

- How to completely classify the period two case?
  - Positions of fixed points - how can we describe this combinatorially?
  - Joint work with Epstein may help describe the new combinatorics.
- Cluster cycles of period ≥ 3.
  - More "secondary maps"
  - Early results suggest far more complexity in the descriptions of the matings
- What about a more general notion of clustering (not restricted to the bicritical case)?
- Parameter space (cf. discussion in Buff, Écalle and Epstein for the parabolic case)?

Thanks for listening!

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