MTH436 - HOMEWORK 5

Solutions to the questions in Section B should be submitted by the start of class on 3/20/19.

A. Warm-up Questions

Question A.1. Compute the following limits.

(i) $\lim_{x\to 0} \frac{\tan x}{x}$.

(ii) $\lim_{x\to 0^+} \frac{1}{x(\ln x)^2}$

(iii) $\lim_{x\to\infty} \left(\frac{1}{x}\right)^x$.

(iv) $\lim_{x\to \frac{\pi}{2}^-}(\sec x - \tan x)$.

Question A.2. Let $f\colon (a,b)\to\mathbb{R}$ be differentiable and suppose that $f''(x)$ exists at $x\in(a,b)$. Prove that

$$f''(x) = \lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

Give an example to show that the limit on the right hand side may exist even if $f''(x)$ does not.

Question A.3. Let f and g be n-times differentiable at x. Prove that

$$(fg)^{(n)}(x) = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}f^{(n-k)}(x)g^{(k)}(x).$$

Question A.4. Suppose $f\colon\mathbb{R}\to\mathbb{R}$ is differentiable. Define $f^{\circ n} = (\overbrace{f\circ f\circ\ldots\circ f}^{n\text{-times}})$. Prove that

$$(f^{\circ n})'(x) = \prod_{k=0}^{n-1} f'(f^{\circ k}(x)) = f'(x)\cdot f'(f(x))\cdots f'(f^{\circ(n-1)}(x))$$

In particular, if we have a finite set $\{x_1, x_2, \ldots x_n\}$ such that $f(x_i) = x_{i+1}$ for $1\le i<n$ and $f(x_n) = x_1$, then $(f^{\circ n})'(x_i) = (f^{\circ n})'(x_j)$ for all $1\le i,j\le n$.

Question A.5. Let $f\colon\mathbb{R}\to\mathbb{R}$ be differentiable. We say x is a fixed point of f if $f(x)=x$.

(i) Prove that if $f'(t)\neq 1$ for all $t\in\mathbb{R}$ then f has at most one fixed point.

(ii) Prove that $g(x) = x+\frac{1}{1+e^x}$ has $0<g'(x)<1$ for all x but g has no fixed point.

Question A.6. Let $f\colon[0,\infty)\to\mathbb{R}$ be differentiable and suppose $\lim_{x\to\infty}(f(x)+f'(x)) = \ell$. Prove that $\lim_{x\to\infty} f(x) = \ell$ and $\lim_{x\to\infty} f'(x) = 0$.

Question A.7. Suppose we know $\lim_{x\to 0}\frac{f(x)}{g(x)} = L$. Is it true that $\lim_{x\to 0}\frac{f'(x)}{g'(x)} = L$? If not, where does the proof break down in the proof of L'Hospital's rule?

B. Submitted Questions

Question B.1. In this question, we investigate an infinitely differentiable function which is not equal to its Taylor series. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x\neq 0,\\ 0 & \text{if } x = 0.\end{cases}$$

(i) Prove that for all $k\in\mathbb{N}$ that $\lim_{y\to\infty}\frac{y^k}{e^{y^2}} = 0$. (Use induction on k and L'Hospital's rule). Convince yourself that $\lim_{y\to-\infty}\frac{y^k}{e^{y^2}} = 0$ (no need to prove this, just check that the same argument will work).

(ii) Deduce that $\lim_{x\to 0}\frac{e^{-1/x^2}}{x^k} = 0$.

(iii) Let $n\in\mathbb{N}\cup\{0\}$. Prove for $x\neq 0$, that $f^{(k)}(x) = f(x)G_k(x)$, where $G_k(x)$ is a rational function.

(iv) Prove by induction and part (ii) that $f^{(n)}(0) = 0$ for all $n\in\mathbb{N}\cup\{0\}$. Observe that f is therefore infinitely differentiable.

(v) Prove that if $R_n(x)$ is the Lagrange form of the remainder from Taylor's theorem and $x\neq 0$ then $\lim_{n\to\infty} R_n(x)\neq 0$.

C. Challenge Questions

Question C.1. Suppose $f\colon [0,\infty) \to \mathbb{R}$ is continuous with $f(0) = 0$ and f is differentiable at x for all $x > 0$. Show that if $f(x)$ is increasing then $\frac{f(x)}{x}$ is increasing for $x \geq 0$.

Question C.2. Suppose $f\colon [-1,1] \to \mathbb{R}$ is three times differentiable with

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Prove that there exists $x \in (-1,1)$ such that $f'''(x) \geq 3$. Show that the equality is obtained if $f(x) = \frac{1}{2}(x^3 + x^2)$.