MTH436 - HOMEWORK 4

Solutions to the questions in Section B should be submitted by the start of class on 3/6/19.

A. Warm-up Questions

Question A.1. Find at which points the following functions are differentiable. and compute the derivatives (if they exist).

(i) $f(x) = x^2 + 1$ if x is rational and $f(x) = 2x$ if x is irrational.
(ii) $f(x) = x|x|$
(iii) $f(x) = \frac{\sin(x^2)}{x}$ for $x \neq 0$ and $f(0) = 0$.
(iv) $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ for $x \neq 0$ and $f(0) = 0$. Prove also that f' is unbounded on the closed interval $[-1, 1]$ and so is not continuous.

Question A.2. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Question A.3. Let I be an interval and $f\colon I \to \mathbb{R}$. Show that if f is differentiable and $f'(x) > 0$ for all $x \in I$ then f is strictly increasing on I.

Question A.4. Suppose $f\colon [0, 1] \to \mathbb{R}$ is uniformly continuous. Is it true that f' is bounded? Prove it or find a counterexample.

Question A.5. Prove that the derivative of an even function is an odd function, and vice versa.

Question A.6. Let $f\colon [0, \infty) \to \mathbb{R}$ be differentiable and suppose that $\lim_{x\to\infty} f'(x) = M$. Find $\lim_{x\to\infty}(f(x + 1) - f(x))$.

Question A.7. Examples from class.

(i) Let $f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(0) = 1$ but f is not increasing in any neighbourhood of 0.
(ii) Let $f(x) = 2x^4 + x^4 \cos\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that 0 is a global minimum for f but for every neighbourhood V of 0 there exists $x, y \in V$ such that $f'(x) > 0$ and $f'(y) < 0$.

B. Submitted Questions

Question B.1. Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Let c be an interior point of I and suppose $\lim_{x\to c} f'(x) = L$. Prove that $f'(c) = L$.

Question B.2. Let $f\colon [0, \infty) \to \mathbb{R}$ be differentiable. Prove or disprove the following.

(i) If $\lim_{x\to\infty} f(x) = L$ then $\lim_{x\to\infty} f'(x) = 0$.
(ii) If $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = M$ then $M = 0$.

C. Challenge Questions

Question C.1. Let $f\colon [0, 2] \to \mathbb{R}$ be continuous on [0,2] and differentiable on $(0, 2)$. Suppose further that $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$. Show there exists $c \in (0, 2)$ such that $f'(c) = \frac{1}{\pi}$.

Question C.2. Let $C^1[0, 1]$ be the set of functions on $[0, 1]$ for which f' is continuous. Show that

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\} + \sup\{|f'(x) - g'(x)| \mid x \in [0, 1]\}$$

defines a metric on $C^1[0, 1]$.

Question C.3. *Path-connectedness* Let X be a metric space. A *path* in X is a continuous map $\gamma\colon [0, 1] \to X$ (where $[0, 1]$ has the standard metric). We say a metric space is *path-connected* if for any $x, y \in X$, there exists a path γ in X such that $\gamma(0) = x$ and $\gamma(1) = y$.

(i) Prove that if γ_1 is a path from a to b and γ_2 is a path from b to c then γ defined as

$$\gamma(x) = \begin{cases} \gamma_1(2x) & x \in [0, \frac{1}{2}], \\ \gamma_2(2x-1) & x \in [\frac{1}{2}, 1]. \end{cases}$$

is a path from a to c.

(ii) Prove that a path is a compact subset of X.

(iii) Prove that if X is a path-connected metric space containing more than one point, then it is uncountable.

(iv) Prove that if X is path-connected, then it is connected.