MTH436 - HOMEWORK 3

Solutions to the questions in Section B should be submitted by the start of class on 2/20/19.

A. Warm-up Questions

Question A.1. Prove that a subset of a metric space equipped with the discrete metric is compact if and only if it is finite.

Question A.2.

(i) Let $f\colon (X, d_1) \to (Y, d_2)$ be a continuous map between metric spaces. Prove that if $K \subseteq X$ is compact then $f(K)$ is compact. (This is an important result)

(ii) Prove that if $f\colon (X, d) \to \mathbb{R}$ is continuous and X is compact then $f(X)$ is bounded and f attains its bounds.

(iii) Formulate a definition for a function $f\colon X \to Y$ between metric spaces to be *uniformly continuous*. Now show that if $f\colon (X, d) \to \mathbb{R}$ is continuous and X is compact then f is uniformly continuous.

Question A.3. Prove that if K_1 and K_2 are compact subsets of a metric space X then $K_1 \cap K_2$ is compact.

Question A.4. *British Rail Metric.* Recall $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is the usual Euclidean norm in $\mathbb{R}^2$. Define a function $d\colon \mathbb{R}^2 \to \mathbb{R}$ as follows. For (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$, we define

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} d_2((x_1, y_1), (x_2, y_2)) & \text{if } (x_2, y_2) = (kx_1, ky_1) \text{ for some } k \in \mathbb{R}, \\ d_2((x_1, y_1), (0, 0)) + d_2((x_2, y_2), (0, 0)) & \text{otherwise.} \end{cases}$$

(i) Prove that d is a metric on $\mathbb{R}^2$.[1] *Hint*: Essentially, if two points lie on the same line through the origin then the distance between them is the usual Euclidean distance. Otherwise, the distance between them is the sum of their distances from the origin. The origin acts as the "hub" through which all journeys must travel.

(ii) Sketch $V_{\frac{1}{2}}((0, 0))$, $V_{\frac{1}{2}}((1, 0))$ and $V_1((3/4, 0))$.

(iii) Does the sequence $x_n = (1/n, 1/n)$ converge? Does the sequence $y_n = (1/n, 1)$ converge?

(iv) Show that the closed unit disk $\overline{D} = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), (0, 0)) \leq 1\}$ is closed and bounded in this metric.

Question A.5. (You may want to use the result of the challenge questions). Prove that if a metric space is compact then it is complete. Is the converse true?

B. Submitted Questions

Question B.1. Let d be the British Rail metric. Consider the closed unit disk $\overline{D} = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), (0, 0)) \leq 1\}$. Is $\overline{D}$ compact? Prove your answer.

Question B.2. Let K_1 and K_2 be compact subsets of a metric space X. Prove that $K_1 \cup K_2$ is compact.

C. Challenge Questions

Sequential compactness. Let (X, d) be a metric space. We say $K \subseteq X$ is *sequentially compact* if every sequence in K has a subsequence which converges to a point of K. We will prove that in a metric space, compactness is equivalent to sequential compactness.

Question C.1. Let K be compact in (X, d) and (x_n) a sequence in K

(i) Prove that if $S = \{x_n \mid n \in \mathbb{N}\}$ is finite, then (x_n) has a convergent subsequence.

[1] Called the British Rail metric. Though the fact that it is possible to get from any point to any other in finite time means it is far more efficient than British rail.

(ii) Assume $S = \{x_n \mid n \in \mathbb{N}\}$ is infinite and assume that S contains no subsequence which converges to a point of K. Prove that for each $x \in K$, there exists $\varepsilon(x) > 0$ such that $V_{\varepsilon(x)}(x) \cap S \subseteq \{x\}$.

(iii) By considering the open cover $\mathcal{U} = \{V_{\varepsilon(x)}(x) \mid x \in K\}$, obtain a contradiction.

Conclude that compactness implies sequential compactness.

Question C.2. Let K be a sequentially compact subset of (X, d). For $\varepsilon > 0$, define an ε-net of K to be a set $S \subseteq K$ such that $K \subseteq \bigcup_{s \in S} V_\varepsilon(s)$. If $\mathcal{U}$ is an open cover of K, we say that $\varepsilon > 0$ is a Lebesgue number for K if for any $x \in K$, there exists $U \in \mathcal{U}$ such that $V_\varepsilon(x) \subseteq U$.

(i) Prove that if $\varepsilon > 0$ and K is sequentially compact, then K has a *finite* ε-net (argue by contradiction).

(ii) Prove that any open cover of a sequentially compact set has a Lebesgue number (again, argue by contradicton).

(iii) Prove that K is compact. Use the fact that any open cover of K has a Lebesgue number ε, and for this $\varepsilon > 0$ it has a corresponding ε-net. Combine these ideas to construct a finite subcover of any open cover of K.

Conclude that sequential compactness implies compactness.