MTH436 - HOMEWORK 2

Solutions to the questions in Section B should be submitted by the start of class on 2/13/19.

A. Warm-up Questions

Question A.1. Suppose I is an interval and $f\colon I \to \mathbb{R}$ is increasing. For $c \in I$, prove the following.

(i) $j_f(c) \geq 0$.
(ii) f is continuous at c if and only if $j_f(c) = 0$.

Question A.2. Let $f\colon [0,1] \to \mathbb{R}$.

(i) Suppose that f takes each value exactly twice. Show that f is not a continuous function.
(ii) Suppose $f(0) < f(1)$ and for each $y \in \mathbb{R}$, the set $f^{-1}(y)$ consists of at most one element. Prove that f is strictly increasing.

Question A.3.

(i) Let (X, d_X) be a metric space and $f, g\colon X \to \mathbb{R}$ be continuous.
   (a) Prove that $f + g$ is continuous.
   (b) Prove that fg is continuous.
(ii) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces and suppose $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous. Prove that $g \circ f$ is continuous.

Question A.4. Find all continuous functions $f\colon (\mathbb{R}, |\cdot|) \to (\mathbb{R}, \delta)$ where δ is the discrete metric. Then find all continuous functions $f\colon (\mathbb{R}, \delta) \to (\mathbb{R}, |\cdot|)$.

Question A.5. Let A be a subset of the metric space X.

(i) Prove that $x \in \operatorname{cl} A$ if and only if for all $\varepsilon > 0$ we have $V_\varepsilon(x) \cap A \neq \varnothing$.
(ii) Prove that $x \in \operatorname{cl} A$ if and only if there exists a sequence in A which converges to x.
(iii) Prove that $x \in b(A)$ if and only if for all $\varepsilon > 0$ we have $V_\varepsilon(x) \cap A \neq \varnothing$ and $V_\varepsilon(x) \cap (X - A) \neq \varnothing$.

Question A.6. Let (X, d) be a metric space. Define the distance between $x \in X$ and $A \subseteq X$ as $\operatorname{dist}(x, A) = \inf\{d(x, a) \mid a \in A\}$. Prove that if $A \subseteq X$ then $\operatorname{cl} A = \{x \in X \mid \operatorname{dist}(x, A) = 0\}$.

B. Submitted Questions

Question B.1. Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous and attains its absolute maximum at $c \in (a, b)$. Show that f is not injective.

Question B.2. Let (X, d) be a metric space.

(i) Prove that $|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y)$ for all $x, y \in X$.
(ii) Prove that $f\colon X \to \mathbb{R}$, $f(x) = \operatorname{dist}(x, A)$ is continuous.

Now suppose A, B are disjoint closed subsets of X. Show that there exists a continuous function $f\colon X \to \mathbb{R}$ satisfying

(a) $0 \leq f(x) \leq 1$ for all $x \in X$.
(b) $f(a) = 0$ for all $a \in A$.
(c) $f(b) = 1$ for all $b \in B$.

(Hint: Try $f(x) = \frac{\operatorname{dist}(x,A)}{\operatorname{dist}(x,A)+\operatorname{dist}(x,B)}$ and check it satisfies the desired properties.)

C. Challenge Questions

Question C.1. Suppose (X, d) is a complete metric space and $Y \subseteq X$. Show that (Y, d) is complete if and only if Y is closed in X.

Question C.2. Let $f\colon (X, d_X) \to (Y, d_Y)$ be a continuous function and A a dense subset of X.

(i) Show that if f is surjctive then $f(A)$ is dense in Y.
(ii) Show that if f is constant on A, then f is constant on X.