# MTH436 - HOMEWORK 1

Solutions to the questions in Section B should be submitted by the start of class on 2/6/19. Remember that you may work together and consult resources, but you should write up your answers on your own and make sure you understand them.

## A. Warm-up Questions

**Question A.1.** Decide if the following functions are uniformly continuous.

(i) $f(x) = \frac{1}{x^2}$ on $[2, \infty)$.
(ii) $f(x) = \sin\left(\frac{1}{x}\right)$ on $(0, 1)$.
(iii) $f(x) = x^2$ on $\mathbb{N}$.

**Question A.2.** Given an example of a function $f\colon [a,b] \cap \mathbb{Q} \to \mathbb{Q}$ which is continuous and bounded but not uniformly continuous. (This is easier than you think!)

**Question A.3.** Suppose $f, g\colon A \to \mathbb{R}$ are Lipschitz functions.

(i) Prove that $f + g$ is Lipschitz.
(ii) Prove that if $f$ and $g$ are bounded functions then $fg$ is Lipschitz.
(iii) Give an example to show that if $f$ or $g$ is not bounded then $fg$ is not necessarily Lipschitz.
(iv Prove the same results with *Lipschitz* replaced by *uniformly continuous*.

**Question A.4.** Suppose $A \subseteq \mathbb{R}$ is bounded and $f\colon A \to \mathbb{R}$ is uniformly continuous. Prove that $f$ is bounded.

**Question A.5.** Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$.

(a) Prove that $f$ is not uniformly continuous.
(b) Suppose $(x_n)$ is a Cauchy sequence in $\mathbb{R}$. Prove that $(g(x_n))$ is a Cauchy sequence.

**Question A.6.** Suppose $f\colon [0, \infty) \to \mathbb{R}$ is continuous $[0, \infty)$ and is uniformly continuous on $[a, \infty)$ for some $a > 0$. Prove that $f$ is uniformly continuous.

**Question A.7.** Let $I$ be an interval. We say that $f\colon I \to \mathbb{R}$ is *absolutely continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{[x_k, y_k] \mid k = 1, \ldots, n\}$ is a finite collection of pairwise disjoint subintervals with $\sum_{k=1}^{n} |x_k - y_k| < \delta$ then $\sum_{k=1}^{n} |f(x_k) - f(y_k)| < \varepsilon$.

(i) Prove that if $f$ is absolutely continuous, it is uniformly continuous.
(ii) Prove that if $f$ is Lipschitz, then it is absolutely continuous.

## B. Submitted Questions

**Question B.1.** Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are uniformly continuous then $g \circ f$ is uniformly continuous.

**Question B.2.** Let $f\colon A \to \mathbb{R}$ be uniformly continuous and $f(x) > k > 0$ for all $x \in A$. Prove that $\frac{1}{f}$ is uniformly continuous.

## C. Challenge Questions

**Question C.1.** Let $f\colon [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that $f$ is uniformly continuous but not absolutely continuous.

**Question C.2.** *The Cantor set.* We will construct a (relatively pathological) closed set in $\mathbb{R}$. We define the *Middle-thirds Cantor set* to be

$$C = \left\{ x \in [0,1] \,\middle|\, x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \text{ where } a_n \in 0, 2 \text{ for all } n \right\}.$$

We can construct $C$ using an iterative method as follows. Let $C_0 = [0, 1]$, and from $C_0$ remove the middle-third interval $(1/3, 2/3)$ to obtain $C_1 = [0, 1/3] \cup [2/3, 1]$. From each of the two intervals making up $C_1$, remove the middle third interval to give the set $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Continue in this way, removing the middle third of each of the $2^n$ intervals of $C_n$ at each step. The eventual set you are left with is in fact $C$: formally $C = \cap_{i=1}^{\infty} C_i$.

(i) Draw the sets $C_n$ for $n = 1, 2, 3, 4$.
(ii) Prove that $C$ is closed.
(iii) What is the sum of the length of all the removed intervals? What does this suggest about the "length" of $C$?
(iv) Prove that $C$ contains no open interval.
(v) Show that $\phi \colon C \to [0, 1]$ given by

$$\phi\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

is a surjection, and deduce that $C$ is uncountable. Does this surprise you with regards to your conclusion in part (iii)?