MTH436 - HOMEWORK 9

No work submitted this week. The sections are to signify approximate difficulty of the problems.

A. Warm-up Questions

Question A.1. For the following sequences f_n, find $\lim_{n\to\infty} f_n$ and decide if this convergence is uniform.

(i) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{n^2+x^2}$.
(ii) $f\colon [0,\infty) \to \mathbb{R}$ given by $f_n(x) = \frac{x^n-1}{x^n+1}$.
(iii) $f_n\colon [-1,1] \to \mathbb{R}$ given by $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$.
(iv) $f_n\colon \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \frac{nx}{1+n^2x^2}$.
(v) $f_n\colon \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = e^{-nx}$.
(vi) $f_n\colon [0,\infty) \to \mathbb{R}$ given by $f_n(x) = \frac{nx}{1+nx}$.

Question A.2. Let $f_n\colon [0,1] \to \mathbb{R}$ given by $f_n(x) = \frac{\cos(nx)}{n}$.

(i) Let $f(x) = 0$ for all $x \in [0,1]$. Prove that $f_n \rightrightarrows f$ on $[0,1]$.
(ii) Compute f_n'. Does $f_n' \to g$ for some function g?

Question A.3. Suppose $f_n\colon A \to \mathbb{R}$ is bounded for each $n \in \mathbb{N}$ (so that for each n there exists $M_n \in \mathbb{R}$ such that $|f_n(x)| \le M_n$ for all $x \in A$). Show that if $f_n \rightrightarrows f$ then f is bounded.

Question A.4. Let $x_1, x_2, \ldots$ be an enumeration of the rationals in $[0,1]$. Define $f_n\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x = x_i \text{ for } 1 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Find f such that $f_n \to f$ and discuss the integrals of f_n and f, and points of continuity of f_n and f.

Question A.5. For the given sequences f_n, decide if the series $\sum f_n$ converges uniformly (here, $a > 0$).

(i) $f_n\colon \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{n^2+x^2}$.
(ii) $f_n\colon [-a,a] \to \mathbb{R}$ given by $f_n(x) = \frac{x^2+n}{x^2+n^3}$.
(iii) $f_n\colon \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \sin(x/n^2)$.

Question A.6. Compute the Radius of convergence for the following power series.

(i) $\sum \frac{2+(-1)^n}{3^n} x^n$.
(ii) $\sum x^{n!}$.
(iii) $\sum a_n x^n$ where

$$a_n = \begin{cases} n & \text{if } n = 3k \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ 0 & \text{if } n = 3k+1 \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ 4^n & \text{if } n = 3k+2 \text{ for some } k \in \mathbb{N} \cup \{0\}. \end{cases}$$

(iv) $\sum a_n x^n$ where $0 < r \le |a_n| \le s$ for all n.
(v) $\sum \frac{\sin(n)}{n} x^n$.

Question A.7. Show that if f is an odd function and $f(x) = \sum a_n x^n$ then $a_{2k} = 0$ for all $k \in \mathbb{N} \cup \{0\}$.

B. (Non-)Submitted Questions

Question B.1. Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{1+nx^2}$.

(i) Find $f = \lim_{n\to\infty} f_n$.
(ii) Show $f_n \rightrightarrows f$.
(iii) Compute $f_n'(x)$. Is it true that $f_n' \to f'$?

Question B.2. Suppose $f_n, g_n \colon A \to \mathbb{R}$ such that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$.

(i) Prove that $(f_n + g_n) \rightrightarrows f + g$.
(ii) Prove that if f_n and g_n are bounded for each $n \in \mathbb{N}$ then $f_n g_n \rightrightarrows fg$.
(iii) Give an example where $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$ but $(f_n g_n)$ does not converge uniformly.

Question B.3. Suppose $f_n \colon A \to \mathbb{R}$ are continuous and $f_n \rightrightarrows f$ and that $x_n \to x$ in A. Prove that $\lim_{n\to\infty} f_n(x_n) \to f(x)$.

C. Challenge Questions

Question C.1. For $x \in \mathbb{R}$, define the *floor* of x to be

$$\lfloor x \rfloor = \sup\{n \in \mathbb{Z} \mid n \leq x\}$$

and the *fractional part* of x to be $\{x\} = x - \lfloor x \rfloor$. Consider $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2}.$$

(i) Find D, the set of discontinuities of f.
(ii) Show that D is countable.
(iii) Show that D is dense in $\mathbb{R}$.
(iv) Show that $f \in \mathcal{R}[a,b]$ for all $-\infty < a < b < \infty$.

Question C.2. Define $s_d \colon [0,1) \colon [0,1)$ by $s_d(x) = \{dx\}$. For $n \in \mathbb{N}$, let $s_d^{\circ n} = s_d \circ s_d \circ \ldots \circ s_d$ be the n-fold composition of s_d.

(i) Show that if $x \in \mathbb{Q}$ then $f^{\circ n}(x)$ is eventually periodic.
(ii) Show that if $x \in R - \mathbb{Q}$ then $\{f^{\circ n}(x) \mid n \in \mathbb{N}\}$ is dense in $[0,1)$.