MTH436 - HOMEWORK 8

Solutions to the questions in Section B should be submitted by the start of class on 4/18/18. Unless specified, you should prove results directly from the definition of the Darboux Integral, not by invoking its equivalence to the Riemann Integral.

A. Warm-up Questions

Question A.1. Define $f\colon [-1,1] \to \mathbb{R}$, $f(x) = \left|x + \frac{1}{2}\right|$. Compute $L(f,\mathcal{P})$ and $U(f,\mathcal{P})$ for the following partitions $\mathcal{P}$.

(i) $\mathcal{P} = (-1, 1)$.
(ii) $\mathcal{P} = (-1, 0, 1)$.
(iii) $\mathcal{P} = (-1,\ -1/2,\ 0,\ 1/4,\ 3/4,\ 1)$.

Question A.2. Let $f_1, f_2\colon [a,b] \to \mathbb{R}$ be bounded.

(i) Show that if $\mathcal{P} \in \mathscr{P}([a,b])$ then $U(f_1 + f_2, \mathcal{P}) \leq U(f_1, \mathcal{P}) + U(f_2, \mathcal{P})$.
(ii) Deduce that $U(f_1 + f_2) \leq U(f_1) + U(f_2)$.
(iii) Give an example where $U(f_1 + f_2) < U(f_1) + U(f_2)$.

Question A.3. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are Darboux Integrable. Show that $f + g$ is Darboux Integrable and that

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

Question A.4. Let $f\colon [a,b] \to \mathbb{R}$ be bounded.

(i) Suppose $k > 0$. Prove that $kU(f) = U(kf)$ and $kL(f) = L(kf)$.
(ii) State and prove an analogous result to the above in the case where $k < 0$.

B. Submitted Questions

Question B.1. Let $f\colon [a,b] \to \mathbb{R}$ be Darboux Integrable and suppose $g\colon [a,b] \to \mathbb{R}$ satisfies $f(x) = g(x)$ for all $x \in [a,b] - E$, where E is a finite set. Show that g is Darboux Integrable and $\int_a^b f = \int_a^b g$.

Question B.2. Let $f\colon [a,b] \to \mathbb{R}$ be bounded.

(i) Show that if f is continuous then it is Darboux Integrable.
(ii) Show that if f is monotone then it is Darboux Integrable.

C. Challenge Questions

Question C.1. *The Riemann-Stieltjes Integral.* Let $\alpha\colon [a,b] \to \mathbb{R}$ be an increasing function and $f\colon [a,b] \to \mathbb{R}$ bounded. Given a partition $\mathcal{P} = (x_0, x_1, \ldots, x_n) \in \mathscr{P}([a,b])$, define

$$U(f,\mathcal{P},\alpha) = \sum_{k=1}^{n} M_k(\alpha(x_i) - \alpha(x_{i-1})),$$
$$U(f,\mathcal{P},\alpha) = \sum_{k=1}^{n} m_k(\alpha(x_i) - \alpha(x_{i-1})).$$

where M_k and m_k are defined as for the Darboux Integral (note that if $\alpha(x) = x$ then this just reduces to the usual Darboux Integral). Then define

$$L_\alpha(f) = \sup\{L(f,\mathcal{P},\alpha) \mid \mathcal{P} \in \mathscr{P}([a,b])\} \quad \text{and} \quad U_\alpha(f) = \inf\{L(f,\mathcal{P},\alpha) \mid \mathcal{P} \in \mathscr{P}([a,b])\}.$$

We say f is Riemann-Stieltjes Integrable (and write $f \in \mathcal{R}(\alpha)$) if $L_\alpha(f) = U_\alpha(f)$. We denote this integral by $\int_a^b f\,\mathrm{d}\alpha$.

(i) Prove the analogue for the Integrability Criterion for this integral.
(ii) Show that if f is continuous or monotone on $[a,b]$ then $f \in \mathcal{R}(\alpha)$.

(iii) Let $a < c < b$ and suppose $\alpha(x) = 0$ for $a \le x < c$ and $\alpha(x) = 1$ for $c \le x \le b$. Prove that
$$\int_a^b f \, \mathrm{d}\alpha = f(c).$$

(iv) Suppose $\alpha' \in \mathcal{R}([a, b])$. Prove that $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}([a, b])$. Furthermore
$$\int_a^b f \, \mathrm{d}\alpha = \int_a^b f(x)\alpha'(x) \, \mathrm{d}x.$$