MTH436 - HOMEWORK 7

Solutions to the questions in Section B should be submitted by the start of class on 4/11/18.

A. Warm-up Questions

Question A.1. Let $f \in \mathcal{R}[a,b]$ and $c \in [a,b]$. Define $F_c(z) = \int_c^z f$ for all $z \in [a,b]$. Find a formula for F_c in terms of F_a.

Question A.2. Calculus time! Let $f \in \mathcal{R}[a,b]$ and define $F(x) = \int_a^x f$ for all $x \in [a,b]$. Find the following in terms of F.

(i) $G(x) = \int_x^b f$.
(ii) $S(x) = \int_a^{x^2} f$.
(iii) $C(x) = \int_a^{\cos x} f$.

Question A.3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and $c > 0$. Define $I_f\colon \mathbb{R} \to \mathbb{R}$ by

$$I_f(x) = \int_{x-c}^{x+c} f$$

Show that I_f is differentiable and compute a formula for $I_f'(x)$.

Question A.4. We will prove the Integration by Substitution formula: If $J = [\alpha, \beta]$ and $\phi : J \to \mathbb{R}$ is continuously differentiable on J, and $f\colon I \to \mathbb{R}$ is continuous with I an interval satisfying $\phi(J) \subseteq I$ then

$$\int_\alpha^\beta f(\phi(t)) \cdot \phi'(t)\,\mathrm{d}t = \int_{\phi(a)}^{\phi(b)} f(x)\,\mathrm{d}x.$$

(i) Let $F(u) = \int_{\phi(a)}^u f(x)\,\mathrm{d}x$ and $G(t) = F(\phi(t))$. Prove that $G'(t) = f(\phi(t)) \cdot \phi'(t)$ for all $t \in J$.
(ii) By setting $t = \beta$, deduce the result.

Question A.5. Let μ^* be outer measure on $\mathbb{R}$.

(i) If $c \in \mathbb{R}$ and $A + c = \{y \in \mathbb{R} \mid y - c \in A\}$ satisfies $\mu^*(A) = \mu^*(A + c)$.
(ii) Show that if $A \subseteq B$ then $\mu^*(A) \le \mu^*(B)$.
(iii) Show that $\mu^*(A \cup B) \le \mu^*(A) + \mu^*(B)$. In particular, show that the union of two null sets is also null.
(iv) Show that if Z_n $(n \in \mathbb{N})$ are null sets then $\bigcup_{n\in\mathbb{N}} Z_n$ is null.

Question A.6. Let $f, g \in \mathcal{R}[a,b]$.

(i) Show that if $t \in \mathbb{R}$ then $\int_a^b (tf + g)^2 \ge 0$
(ii) Show that if $t > 0$ then $2\left|\int_a^b fg\right| \le t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$.

B. Submitted Questions

Question B.1. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and suppose that $\int_a^x f = \int_x^b f$ for all $x \in [a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Question B.2. Let $f, g \in \mathcal{R}[a,b]$.

(i) Prove that if $\int_a^b f^2 = 0$ then $\int_a^b fg = 0$.
(ii) Prove that

$$\left|\int_a^b fg\right|^2 \le \left(\int_a^b |fg|\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

C. Challenge Questions

Question C.1. Let $f, g, h\colon [a, b] \to \mathbb{R}$ be continuous. Define $(f, g) = \int_0^1 fg$.

(i) Prove that $(f, g) = (g, f)$.
(ii) Prove that $(\alpha f, g) = \alpha(f, g)$ for all $\alpha \in \mathbb{R}$.
(iii) Prove that $(f + h, g) = (f, g) + (h, g)$.
(iv) Prove that $(f, f) = 0$ if and only if $f(x) = 0$ for all $x \in [0, 1]$.

Then prove that $\|f\| = \sqrt{(f, f)}$ defines a norm on the space of continuous functions on $[0, 1]$. Also show that we have

$$\|f\|^2 + \|g\|^2 = \frac{\|x + y\|^2 + \|x - y\|^2}{2}$$

Question C.2. We will show that outer measure is not additive. That is, it is not necessarily true that if A and B are disjoint then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. We assume to the contrary.

(i) For $x, y \in [0, 1]$, let $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Prove that $\sim$ is an equivalence relation on $[0, 1]$.
(ii) For each equivalence class X of $\sim$, choose a representative $x \in X$. Let $A \subseteq [0, 1]$ be the set of all these representatives. Show that if $q, r \in \mathbb{Q}$ then $(A + q) \cap (A + r) = \varnothing$.
(iii) Let $Y = \mathbb{Q} \cap [-1, 1]$. Show that

$$[0, 1] \subseteq U := \bigcap_{y \in Y} (A + y) \subseteq [-1, 2].$$

(iv) Show that if $\mu^*(A) = 0$ then $\mu^*(U) = 0$.
(v) Show that if $\mu^*(A) > 0$ then $\mu^*(U) = \infty$.
(vi) Deduce from the previous two parts that outer measure cannot be additive.