MTH436 - HOMEWORK 6

Solutions to the questions in Section B should be submitted by the start of class on 4/4/18.

A. Warm-up Questions

Question A.1. Suppose $f\colon [a,b] \to \mathbb{R}$ has the property that $f \in \mathcal{R}[c,b]$ for all $c \in (a,b)$. Is it true that $f \in \mathcal{R}[a,b]$? What if f is bounded?

Question A.2. Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

Is $f \in \mathcal{R}[0,1]$? Justify your answer.

Question A.3. Give an example of $f \in \mathcal{R}[0,1]$ for which there is no $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a)$$

Question A.4. Let $a > 0$ and suppose $f \in \mathcal{R}[-a,a]$.

(i) Show that if f is even then $\int_{-a}^a f = 2\int_0^a f$.

(ii) Show that if f is odd then $\int_{-a}^a f = 0$.

Question A.5. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and $f(x) \geq 0$ for all $x \in [a,b]$. Show that if $\int_a^b f = 0$ then $f(x) = 0$ for all $x \in [a,b]$.

B. Submitted Questions

Question B.1.

(i) Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

(ii) Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous and $g(x) > 0$ for all $x \in [a,b]$. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b fg = f(c)\int_a^b g.$$

Question B.2. Suppose $f\colon [-a^2, a^2] \to \mathbb{R}$ is continuous. Prove that

$$\int_{-a}^a f\left(x^2\right)\,\mathrm{d}x = 2\int_0^a f\left(x^2\right)\,\mathrm{d}x.$$

C. Challenge Questions

Question C.1. Let $f\colon [a,b] \to \mathbb{R}$ be continuous with $f(x) \geq 0$ for all $x \in [a,b]$. For $p \in \mathbb{N}$, define $\|f\|_p = \left(\int_a^b f^p\right)^{1/p}$ to be the p-norm of f. Prove that if we define $\|f\|_\infty = \sup\{f(x) \mid x \in [a,b]\}$ then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Question C.2. Using the notation from the previous questions, show that for $p = 1, 2$ and ∞, the function $d_p(f,g) = \|f-g\|_p$ defines a metric on the set $C([a,b])$ of continuous functions on $[a,b]$.