MTH436 - HOMEWORK 5

Solutions to the questions in Section B should be submitted by the start of class on 3/21/18.

A. Warm-up Questions

Question A.1. Let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = x^2$. For the following tagged partitions $\dot{\mathcal{P}}$, compute $S(f; \dot{\mathcal{P}})$.

(i) $\dot{\mathcal{P}} = \left\{ \left(\left[0, \frac{1}{2}\right], 0 \right), \left(\left[\frac{1}{2}, 1\right], \frac{1}{2} \right) \right\}$
(ii) $\dot{\mathcal{P}} = \left\{ \left(\left[0, \frac{1}{2}\right], \frac{1}{4} \right), \left(\left[\frac{1}{2}, 1\right], \frac{3}{4} \right) \right\}$
(iii) $\dot{\mathcal{P}} = \left\{ \left(\left[0, \frac{1}{2}\right], \frac{1}{2} \right), \left(\left[\frac{1}{2}, 1\right], 1 \right) \right\}$
(iv) $\dot{\mathcal{P}} = \left\{ \left(\left[\frac{k-1}{n}, \frac{k}{n}\right], \frac{k-1}{n} \right) \middle| k = 1, 2, \ldots, n \right\}$
(v) $\dot{\mathcal{P}} = \left\{ \left(\left[\frac{k-1}{n}, \frac{k}{n}\right], \frac{2k-1}{2n} \right) \middle| k = 1, 2, \ldots, n \right\}$
(vi) $\dot{\mathcal{P}} = \left\{ \left(\left[\frac{k-1}{n}, \frac{k}{n}\right], \frac{k}{n} \right) \middle| k = 1, 2, \ldots, n \right\}$

Question A.2. Suppose $f \in \mathcal{R}[a,b]$. Let $(\dot{\mathcal{P}}_n)$ be a sequence of tagged partitions such that $\|\dot{\mathcal{P}}_n\| \to 0$. Prove that $\int_a^b f = \lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n)$.

Question A.3. Let $\dot{\mathcal{P}} = \{(I_k = [x_{k-1}, x_k], t_k)\}_{k=1}^n$ be a tagged partition of $[0,3]$ with $\|\dot{\mathcal{P}}\| < \delta$. Define

$$\dot{\mathcal{P}}_1 = \{I_k \in \mathcal{P} \mid t_k \in [0,1]\}$$
$$\dot{\mathcal{P}}_2 = \{I_k \in \mathcal{P} \mid t_k \in [1,2]\}.$$

Prove the following.

(i) $[0, 1-\delta] \subseteq \bigcup_{I_k \in \dot{\mathcal{P}}_1} I_k \subseteq [0, 1+\delta]$.
(ii) $[1+\delta, 2-\delta] \subseteq \bigcup_{I_k \in \dot{\mathcal{P}}_2} I_k \subseteq [1-\delta, 2+\delta]$.

Question A.4. Suppose $f \in \mathcal{R}[a,b]$ and $|f(x)| \leq M$ for all $x \in [a,b]$. Prove that

$$\left| \int_a^b f \right| \leq M(b-a)$$

Question A.5. Show the Dirichlet function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not Riemann Integrable.

B. Submitted Questions

Question B.1. Let $f\colon [a,b] \to \mathbb{R}$ is bounded. Suppose there exists two sequences of tagged partitions $(\dot{\mathcal{P}}_n)$ and $(\dot{\mathcal{Q}}_n)$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\|\dot{\mathcal{Q}}_n\| \to 0$ but $\lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n) \neq \lim_{n\to\infty} S(f; \dot{\mathcal{Q}}_n)$. Show that $f \notin \mathcal{R}[a,b]$.

Question B.2. Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Is $f \in \mathcal{R}[0,1]$? Justify your answer.

C. Challenge Questions

Question C.1. Let $C[0,1]$ be the set of continuous functions on $[0,1]$. For $f, g \in C[0,1]$, define $d(f,g) = \int_0^1 |f-g|$. Prove that d is a metric on $C[0,1]$[1]. Is this metric space complete?

Question C.2. Prove that $\mathcal{R}[0,1]$ and $C[0,1]$ are (real) vector spaces[2]. Furthermore, show that $\|f\|_1 = \int_0^1 |f|$ defines a norm[3] on C[0,1]. Does it define a norm on $\mathcal{R}[0,1]$?

[1]You will need to assume that $C[0,1] \subseteq \mathcal{R}[0,1]$

[2]Come ask me for the definition of a vector space if you haven't seen it before

[3]Again, you can ask me if you haven't seen the definition before