MTH436 - HOMEWORK 4

Solutions to the questions in Section B should be submitted by the start of class on 2/28/18.

A. Warm-up Questions

Question A.1. Compute the following limits.

(i) $\lim_{x\to 0} \frac{\tan x}{x}$.

(ii) $\lim_{x\to 0^+} \frac{1}{x(\ln x)^2}$

(iii) $\lim_{x\to\infty} \left(\frac{1}{x}\right)^x$.

(iv) $\lim_{x\to\frac{\pi}{2}^-} (\sec x - \tan x)$.

Question A.2. Let $f\colon (a,b) \to \mathbb{R}$ be differentiable and suppose that $f''(x)$ exists at $x \in (a,b)$. Prove that

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Give an example to show that the limit on the right hand side may exist even if $f''(x)$ does not.

Question A.3. Let f and g be n-times differentiable at x. Prove that

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)}(x) g^{(k)}(x).$$

Question A.4. Suppose $f\colon \mathbb{R}\to\mathbb{R}$ is differentiable. Define $f^{\circ n} = (\overbrace{f\circ f\circ \ldots \circ f}^{n\text{-times}})$. Prove that

$$f^{\circ n}(x) = \prod_{k=0}^{n-1} f'(f^{\circ k}(x)) = f'(x)\cdot f'(f(x))\cdots f'(f^{\circ(n-1)}(x))$$

In particular, if we have a finite set $\{x_1, x_2, \ldots x_n\}$ such that $f(x_i) = x_{i+1}$ for $1 \le i < n$ and $f(x_n) = x_1$, then $f^{\circ n}(x_i) = f^{\circ n}(x_j)$ for all $1 \le i, j \le n$.

Question A.5. Let $f\colon [0,\infty) \to \mathbb{R}$ be differentiable and suppose $\lim_{x\to\infty}(f(x) + f'(x)) = \ell$. Prove that $\lim_{x\to\infty} f(x) = \ell$ and $\lim_{x\to\infty} f'(x) = 0$.

B. Submitted Questions

Question B.1. Let $f\colon \mathbb{R}\to\mathbb{R}$ be differentiable. We say x is a fixed point of f if $f(x) = x$.

(i) Prove that if $f'(t) \neq 1$ for all $t \in \mathbb{R}$ then f has at most one fixed point.

(ii) Prove that $g(x) = x + \frac{1}{1+e^x}$ has $0 < g'(x) < 1$ for all x but g has no fixed point.

Question B.2. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(i) Prove that for all $k \in \mathbb{N}$ we have $\lim_{x\to 0} \frac{f(x)}{x^k} = 0$.

(ii) Compute $f^{(n)}(0)$ for all $n \in \mathbb{N}$ and so write out the nth Taylor polynomial for f at 0.

(iii) Prove that $\lim_{n\to\infty} R_n(x) \neq 0$ for all $x \neq 0$.

C. Challenge Questions

Question C.1. Suppose $f\colon [0,\infty) \to \mathbb{R}$ is continuous with $f(0) = 0$ and f is differentiable at x for all $x > 0$. Show that if $f(x)$ is increasing then $\frac{f(x)}{x}$ is increasing for $x \ge 0$.

Question C.2. Suppose $f\colon [-1,1] \to \mathbb{R}$ is three times differentiable with

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Prove that there exists $x \in (-1,1)$ such that $f'''(x) \ge 3$. Show that the equality is obtained if $f(x) = \frac{1}{2}(x^3 + x^2)$.