MTH436 - HOMEWORK 1

Solutions to the questions in Section B should be submitted by the start of class on 2/7/18.

A. Warm-up Questions

Question A.1. Decide if the following functions are uniformly continuous.

(i) $f(x) = \frac{1}{x^2}$ on $[2, \infty)$.
(ii) $f(x) = \sin\left(\frac{1}{x}\right)$ on $(0, 1)$.
(iii) $f(x) = x^2$ on $\mathbb{N}$.

Question A.2. Given an example of a function $f\colon [a, b] \cap \mathbb{Q} \to \mathbb{R}$ which is continuous and bounded but not uniformly continuous.

Question A.3. Suppose $f, g\colon A \to \mathbb{R}$ are Lipschitz functions.

(i) Prove that $f + g$ is Lipschitz.
(ii) Prove that if f and g are bounded functions then fg is Lipschitz.
(iii) Give an example to show that fg is not necessarily Lipschitz.

Question A.4. Suppose $A \subseteq \mathbb{R}$ is bounded and $f\colon A \to \mathbb{R}$ is uniformly continuous. Prove that f is bounded.

Question A.5. Suppose I is an interval and $f\colon I \to \mathbb{R}$ is increasing. For $c \in I$, prove the following.

(i) $j_f(c) \geq 0$.
(ii) f is continuous at c if and only if $j_f(c) = 0$.

Question A.6. Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous and attains its absolute maximum at $c \in (a, b)$. Show that f is not injective.

B. Submitted Questions

Question B.1. Let I be an interval. We say that $f\colon I \to \mathbb{R}$ is *absolutely continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{[x_k, y_k] \mid k = 1, \ldots, n\}$ is a finite collection of pairwise disjoint subintervals with $\sum_{k=1}^{n} |x_k - y_k| < \delta$ then $\sum_{k=1}^{n} |f(x_k) - f(y_k)| < \varepsilon$.

(i) Prove that if f is absolutely continuous, it is uniformly continuous.
(ii) Prove that if f is Lipschitz, then it is absolutely continuous.

Question B.2. Let $f\colon [0, 1] \to \mathbb{R}$.

(i) Suppose that f takes each value exactly twice. Show that f is not a continuous function.
(ii) Suppose $f(0) < f(1)$ and for each $y \in \mathbb{R}$, the set $f^{-1}(y)$ consists of at most one element. Prove that if f is continuous then f is strictly increasing.

C. Challenge Questions

Question C.1. Let $f\colon [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is uniformly continuous but not absolutely continuous.

Question C.2. *The Cantor set.* We will construct a (relatively pathological) closed set in $\mathbb{R}$. We define the *Middle-thirds Cantor set* to be

$$C = \left\{ x \in [0, 1] \,\middle|\, x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \text{ where } a_n \in 0, 2 \text{ for all } n \right\}.$$

We can construct C using an iterative method as follows. Let $C_0 = [0, 1]$, and from C_0 remove the middle-third interval $(1/3, 2/3)$ to obtain $C_1 = [0, 1/3] \cup [2/3, 1]$. From each of the two intervals making up C_1, remove the middle third interval to give the set $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Continue in this way, removing the middle third of each of the 2^n intervals of C_n at each step. The eventual set you are left with is in fact C: formally $C = \cap_{i=1}^{\infty} C_i$.

(i) Draw the sets C_n for $n = 1, 2, 3, 4$.
(ii) Prove that C is closed.
(iii) What is the sum of the length of all the removed intervals? What does this suggest about the "length" of C?
(iv) Prove that C contains no open interval.
(v) Show that $\phi\colon C \to [0, 1]$ given by

$$\phi\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

is a surjection, and deduce that C is uncountable. Does this surprise you with regards to your conclusion in part (iii)?