MTH435 - HOMEWORK 9

Solutions to the questions in Section B should be submitted by the start of class on 12/05/18.

A. Warm-up Questions

Question A.1. Compute the following limits, if they exist, using the definition. If you feel like reliving the good days of calculus, also use the limit laws in the appropriate cases.

(a) $\lim_{x\to 1} \frac{x}{x+1}$

(b) $\lim_{x\to +\infty} \frac{\sqrt{x}-2}{\sqrt{x}+3}$

(c) $\lim_{x\to 1} \frac{x^2-3x+2}{x^2-1}$

(d) $\lim_{x\to 0} x\operatorname{sgn}(x)$

(e) $\lim_{x\to 0} \frac{\sqrt{x+3}}{x}$

(f) $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R}-\mathbb{Q}. \end{cases}$$

Question A.2. Let $A \subseteq \mathbb{R}$, $f, g\colon A \to \mathbb{R}$ and c a cluster point of A. Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ and $b \in \mathbb{R}$. Prove the following. For a bit more of a challenge, try to prove them directly from the definitions.

(a) $\lim_{x\to c} (f+g)(x) = L+M$.

(b) $\lim_{x\to c} (fg)(x) = LM$.

(c) $\lim_{x\to c} (bf)(x) = bL$.

(d) If $M \neq 0$ and $g(x) \neq 0$ for all $x \in A$ then $\lim_{x\to c} \left(\frac{f}{g}\right)(x) = \frac{L}{M}$.

Also prove that

$$\lim_{x\to c} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$$

Question A.3. Let $A \subseteq \mathbb{R}$, $f_i\colon A \to \mathbb{R}$ for $i = 1, \ldots, k$ and c a cluster point of c. Prove that

$$\lim_{x\to c} (f_1 + f_2 + \cdots + f_k)(x) = \sum_{i=1}^{k} \lim_{x\to c} f_i(x).$$

Question A.4. $A \subseteq \mathbb{R}$. Prove the following.

(a) Let $f, g, h\colon A \to \mathbb{R}$ and c be a cluster point of A. If $f(x) \le g(x) \le h(x)$ for all $x \in A - \{c\}$ and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

(b) Let $f\colon A \to \mathbb{R}$ and c a cluster point of $A \cap (c, \infty)$. Then the following are equivalent.

  (i) $\lim_{x\to c^+} f(x) = L$

  (ii) For all sequences (x_n) in $A \cap (c,\infty)$ with $x_n \to c$ we have $f(x_n) \to L$.

Question A.5. Let $f\colon \mathbb{R} \to \mathbb{R}$ and suppose $\lim_{x\to 0} f(x) = L$. Define $g(x) = f(x-c)$. Prove that $\lim_{x\to c} g(x) = L$.

Question A.6. Let $A \subseteq \mathbb{R}$, $f\colon A \to \mathbb{R}$ and c a cluster point of both $A \cap (c,\infty)$ and $A \cap (-\infty, c)$. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$.

Question A.7. Let $A \subseteq \mathbb{R}$, $f\colon A \to \mathbb{R}$ and c a cluster point of A. Supose that $\lim_{x\to c} f(x) = L > 0$ and $\lim_{x\to c} g(x) = +\infty$. Prove that $\lim_{x\to c} f(x)g(x) = +\infty$. What happens if $L = 0$?

Question A.8. Suppose $f\colon (0,\infty) \to \mathbb{R}$. Show that $\lim_{x\to\infty} f(x) = +\infty$ if and only if $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = \infty$.

B. Submitted Questions

Question B.1. let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Find all points c such that $\lim_{x\to c} f(x)$ exists. Also prove that the limit does not exist at the other points.

Question B.2. Let $f, g\colon (a, \infty)$ and suppose $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} g(x) = \infty$. Prove that $\lim\limits_{x\to\infty} (f \circ g)(x) = L$.

C. Challenge Questions

Question C.1. *Hausdorff metric.* Let $X = \{V \subseteq \mathbb{R} \mid V \neq \varnothing, V \text{ is closed and bounded}\}$. Prove that $d\colon X \times X \to \mathbb{R}$ defined by

$$d(U, V) = \max\{\sup_{u\in U} \inf_{v\in V} |u - v|, \sup_{v\in V} \inf_{u\in U} |u - v|\}$$

is a metric on X.

Question C.2. *The p-adic metric.* This problem requires a little number theory. Let p be a prime number (at the first run through, perhaps fix $p = 2$ or $p = 3$ to see what is going on).

(a) Prove that if $n \in \mathbb{Z}$, then n has a unique expansion in base p; that is

$$n = \sum_{i=0}^{k} a_i p^i$$

where $a_i \in \{0, 1, \ldots, p-1\}$ for each i. In particular, show that for an integer n, there exists a unique $r \geq 0$ such that $n = p^r a$, where p does not divide a.

(b) Deduce that if $0 \neq x \in \mathbb{Q}$, then there exists a unique integer r such that $x = p^r \frac{a}{b}$, where neither a nor b are divisible by p.

(c) Show that if x is as above, then if we define $|x|_p = p^{-r}$ and $|0|_p = 0$, then $d_p(x, y) = |x - y|_p$ defines a metric on $\mathbb{Q}$.

(d) Show that $(\mathbb{Q}, |\cdot|_p)$ is not complete.

(e) (Advanced) Following the method of Question C2 of Homework 5, construct a completion $\mathbb{Q}_p$ of $\mathbb{Q}$ in this metric. Observe that this completion is not an ordered field, and it does not satisfy the Archimidean property.

(f) Ponder how one would do analysis on such a space.