MTH435 - HOMEWORK 8

Solutions to the questions in Section B should be submitted by the start of class on 11/14/18. In this homework, the metrics d_1, d_2 and d_∞ on $\mathbb{R}^2$ are defined as follows.

$$d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$
$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$$
$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

On any non-empty set X, the discrete metric δ is defined as

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

A. Warm-up Questions

Question A.1. Draw $V_1((0,0))$ in $\mathbb{R}^2$ for the metrics d_1, d_2, d_∞ and δ.

Question A.2. Let (X, d) be a metric space.

(a) Show that if $x, y \in X$, $r \in \mathbb{R}$ and $d(x, y) > 2r$ then $V_r(x)$ and $V_r(y)$ are disjoint.
(b) Show that if $x \neq y$ in X then there exist neighbourhoods U of x and V of y such that $U \cap V = \varnothing$.[1]
(c) Deduce that, for any metric space X containing more than one point, it is not possible that X and $\varnothing$ are the only open subsets of X.

Question A.3. Let $X = [0, 1]$ with the usual absolute value as a metric. Is the set $U = \left[0, \frac{1}{2}\right)$ open? Is it closed?

Question A.4. Prove that if X is a discrete metric space and (x_n) is a Cauchy sequence then (x_n) is eventually constant.

Question A.5. Prove that the metrics d_1, d_2 and d_∞ on $\mathbb{R}^2$ are topologically equivalent. In fact, prove that

$$\frac{1}{2} d_1(a, b) \leq \frac{1}{\sqrt{2}} d_2(a, b) \leq d_\infty(a, b) \leq d_2(a, b) \leq d_1(a, b).$$

for all $a = (x_1, y_1)$, $b = (x_2, y_2)$ in $\mathbb{R}^2$.

Question A.6. Using the notation from class, prove that if d and d' are Lipschitz equivalent metrics, then there exists $K > 0$ such that for all $x \in X$ we have

$$V'_{\frac{\varepsilon}{K}}(x) \subseteq V_\varepsilon(x) \subseteq V'_{K\varepsilon}(x).$$

Question A.7. Suppose that d and d' are equivalent metrics on a set X. Prove that (X, d) is complete if and only if (X, d') is complete,

B. Submitted Questions

Question B.1. Prove that $\mathbb{R}^2$ equipped with d_∞ is a complete metric space.

Question B.2. Consider the metrics $d(x, y) = |x - y|$ and $d'(x, y) = \min\{1, |x - y|\}$.

(a) Prove that d and d' are topologically equivalent.
(b) Prove that d and d' are not Lipschitz equivalent.

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[1] This is called the *Hausdorff* property. You can remember the definition by noting that it says x and y can be "housed off" from each other.

C. Challenge Questions

Question C.1. It was shown in class that one of the consequences of the Completeness Property of $\mathbb{R}$ is that all Cauchy sequences converge. In this question, we will show that if we assume all Cauchy sequences converge *and* $\mathbb{R}$ has the Archimedean Property then every non-empty set in $\mathbb{R}$ which is bounded above has a least upper bound. This will prove that our definition of completeness of a metric space is equivalent to our original definition of completeness in $\mathbb{R}$.[2]

(i) Prove that if (x_n) is an increasing sequence which is bounded above, then it is Cauchy. Deduce that all bounded monotonic sequences converge.

(ii) Let A be a non-empty set which is bounded above. Construct a Cauchy sequence (a_n) in A as follows.
  - Let $a_1 \in A$ and u_1 be an upper bound of A. Define $p_1 = \frac{1}{2}(a_1 + u_1)$.
  - If p_1 is an upper bound of A, define $a_2 = a_1$ and $u_2 = p_1$.
  - If p_1 is not an upper bound of A, choose $a_2 \in A$ such that $a_2 > p_1$. Then define $u_2 = u_1$.
  - Continue inductively, defining $p_{k+1} = \frac{1}{2}(a_k + u_k)$. Deduce that (a_k) are (u_k) are monotonic sequences and thus converge.

(iii) Prove that $\sup A = \lim_{n\to\infty} a_n = \lim_{n\to\infty} u_n$.

[2] Unless you assume $\mathbb{R}$ does not satisfy the Archimedean property, which leads to topics such as Non-standard Analysis and p-adic Analysis.