MTH435 - HOMEWORK 6

Solutions to the questions in Section B should be submitted by the start of class on 11/7/18.

A. Warm-up Questions

Question A.1. Prove the following assertions which strengthen the Root and Ratio tests.

(a) If $\limsup |x_n|^{\frac{1}{n}} = r$ and $r < 1$ then $\sum x_n$ converges absolutely.

(b) If $\limsup \left|\frac{x_{n+1}}{x_n}\right| = r$ and $r < 1$ then $\sum x_n$ converges absolutely.

Question A.2. Discuss the convergence or divergence of $\sum x_n$ for the following sequences.

(a) $x_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.

(b) $x_n = \frac{1}{n \log n}$.

(c) $x_n = \frac{n}{2^n}$.

(d) $x_n = \frac{n^n}{e^n}$.

(e) $x_{2n-1} = \frac{1}{(2n-1)^2}$ and $x_{2n} = \frac{1}{(2n)^3}$.

(f) $x_n = \frac{n!}{n^n}$.

Question A.3. Prove the following are metric spaces.

(a) $d(x, y) = \frac{|x-y|}{1+|x-y|}$ on $\mathbb{R}$.

(b) $d(x, y) = \min\{1, |x - y|\}$ on $\mathbb{R}$.

(c) $d((x_1, y_1),(x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ on $\mathbb{R}^2$.

(d) $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$ on an arbitrary non-empty set X.

(e) $d(x, y) = |x - y|$ on $\mathbb{Q}$.

(f) $d((x_1, y_1),(x_2, y_2), y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ on $\mathbb{R}^2$.

Question A.4. Show that the following are *not* metric spaces.

(a) $d(x, y) = |x - y|^2$ on $\mathbb{R}$.

(b) $d((x_n),(y_n)) = |\lim_{n\to\infty}(x_n - y_n)|$ on the set of convergent sequences.

(c) $d((x_n),(y_n)) = \sup\{|x_n - y_n| \mid n \in \mathbb{N}\}$ on the set of sequences of real numbers.

(d) $d(X, Y) = \inf\{|x - y| \mid x \in X, y \in Y\}$ on the set of non-empty subsets of $\mathbb{R}$.

B. Submitted Questions

Question B.1. Suppose $a_n > 0$ for all n. By using the following steps (or otherwise, if you find a better method!), prove that if $\sum \frac{a_n}{1+a_n}$ converges then $\sum a_n$ converges.

(a) Show that if $\sum \frac{a_n}{1+a_n}$ converges then there exists K such that if $n \geq K$ then $0 < a_n < 1$.

(b) Deduce that $n \geq K$ implies $\frac{a_n}{2} \leq \frac{a_n}{1+a_n}$.

(c) Prove that $\sum a_n$ converges.

Question B.2. Let X be the set of bounded sequences of real numbers. Define $d\colon X \times X \to \mathbb{R}$ by $d((x_n),(y_n)) = \sup\{|x_n - y_n| \mid n \in \mathbb{N}\}$. Prove that (X, d) is a metric space.

C. Challenge Questions

Question C.1. Let X be the set of sequences (x_n) for which the associated series $\sum x_n$ is absolutely convergent and define $d\colon X \times X \to \mathbb{R}$ by $d((x_n),(y_n)) = \sum |x_n - y_n|$. Prove that (X, d) is a metric space.

Question C.2. Let X be a space with a semi-metric d (that is, d satisfies all the properties of a metric except it is not necessarily true that $d(x, y) = 0$ implies $x = y$.)

(i) Prove that the relation $\sim$ on X defined by $x \sim y$ if and only if $d(x, y) = 0$ is an equivalence relation on X.

(ii) Let $X/{\sim}$ be the set of equivalence classes of $\sim$. Prove that $(X/{\sim}, d')$ is a metric space, where $d'([x],[y]) = d(x, y)$ (here, $[x]$ denotes the equivalence class of x under $\sim$ - you will have to check d' is well-defined).