MTH435 - HOMEWORK 6

Solutions to the questions in Section B should be submitted by the start of class on 10/31/18.

A. Warm-up Questions

Question A.1. Decide if the following sequences are properly divergent.

(a) $(\sqrt{n})$.
(b) $(n \cos n)$
(c) $\left(\frac{2n}{\sqrt{n+7}}\right)$
(d) $\left(\frac{n+1}{\sqrt{n^2+5}}\right)$

Question A.2. Compute the following.

(a) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$.
(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.
(c) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{3n}$.

Question A.3. Give examples of the following.

(a) A properly divergent sequence that is not monotonic.
(b) An unbounded sequence that is not properly divergent.
(c) An unbounded sequence with a convergent subsequence.

Question A.4. Let (x_n) be a sequence of non-zero real numbers. Prove or disprove the following.

(a) If $x_n \to 0$ then $\frac{1}{x_n}$ is properly divergent.
(b) If $x_n \to +\infty$ then $\frac{1}{x_n} \to 0$.

Question A.5. Prove or disprove the following.

(a) If $x_n \to +\infty$ and (y_n) is convergent then $(x_n + y_n) \to +\infty$.
(b) If $x_n \to +\infty$ and (y_n) is convergent then $(x_n y_n) \to +\infty$.
(c) If $x_n \to +\infty$ and $(x_n y_n)$ converges then (y_n) converges.
(d) If $x_n \to +\infty$ and $y_n \to +\infty$ then $(x_n - y_n)$ converges.
(e) If $\sum a_n$ converges then $\sum a_n^2$ converges. What if $a_n > 0$ for all n?
(f) If $\sum a_n$ converges and $a_n > 0$ for all n then $\sum \sqrt{a_n}$ converges.
(g) If (a_n) is a decreasing sequence of positive numbers and if $\sum a_n$ converges then $\lim_{n\to\infty} n a_n = 0$.
(h) If (a_n) is such that $(n^2 a_n)$ converges then $\sum a_n$ is absolutely convergent.

B. Submitted Questions

Question B.1. Let (x_n) and (y_n) be sequences of positive numbers and suppose $\left(\frac{x_n}{y_n}\right) \to +\infty$.

(a) Show that if $y_n \to +\infty$ then $x_n \to +\infty$.
(b) Show that if (x_n) is a bounded sequence then (y_n) converges.

Question B.2.

(a) Show that if $\sum a_n$ is absolutely convergent and (b_n) is a bounded sequence then $\sum a_n b_n$ is absolutely convergent.
(b) Give an example of a conditionally convergent series $\sum a_n$ and a bounded sequence (b_n) such that $\sum a_n b_n$ is divergent.

C. Challenge Questions

Question C.1. *Another definition of e.* We give an alternative definition of e and prove it is irrational. First, consider the sequence (e_n) defined by

$$e_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

(i) Define $s_n = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$ for $n \geq 1$. Prove that $s_n \geq e_n$ for all n and deduce that (e_n) converges. We denote this limit by e.

(ii) Assume $e = p/q$ in lowest terms and show that $e - e_q = \frac{k}{q!}$ for some $k \in \mathbb{N}$.

(iii) Prove that $e - e_q < \frac{1}{q!} \sum_{k=1}^{\infty} 2^{-k}$ and obtain a contradiction to part (ii). Conclude that e is irrational.

Question C.2. We now prove that the new definition of e given in C.1. is equivalent to the definition given in class. Denote

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

(i) Use the binomial theorem to prove that

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{(k-1)}{n}\right) \leq \sum_{k=0}^{n} \frac{1}{k!}$$

and so deduce $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \leq e$.

(ii) Use the binomial theorem to show that for all $n, m \in \mathbb{N}$ we have

$$\left(1 + \frac{1}{n}\right)^{m+n} = \sum_{k=0}^{m+n} \frac{1}{k!} \frac{(n+m)!}{(n+m-k)!} \frac{1}{n^k} \geq \sum_{k=0}^{m} \frac{1}{k!} \frac{(n+m)!}{(n+m-k)!} \frac{1}{n^k}.$$

Deduce that $\left(1 + \frac{1}{n}\right)^{m+n} \geq \sum_{k=0}^{m} \frac{1}{k!}$.

(iv) Using the result of part (iii), show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \geq \sum_{k=0}^{m} \frac{1}{k!}$. Then, taking the limit as $m \to \infty$, conclude that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \geq e$.

(v) Use the results of parts (i) and (iv) to prove that $\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$.