MTH435 - HOMEWORK 5

Solutions to the questions in Section B should be submitted by the start of class on 10/17/18.

A. Warm-up Questions

Question A.1. Using the definition, decide whether or not the following sequences are Cauchy: $x_n = \frac{1}{n(n+1)}$, $y_n = \sqrt{n}$ and $z_n = \frac{n-1}{n}$.

Question A.2. Let (x_n) and (y_n) be Cauchy sequences. *Directly from the definition...*

(a) Prove that $(x_n + y_n)$ is Cauchy.
(b) Prove that $(x_n y_n)$ is Cauchy.
(c) Prove that $|x_n - y_n|$ converges.
(d) Prove that if $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$ then (x_n) is eventually constant.

Question A.3. Prove or disprove the following for a sequence (x_n).

(a) (x_n) is Cauchy if and only if $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$.
(b) (x_n) is contractive if and only if it is Cauchy.

Question A.4. Let $x_1 = 5$ and $x_{n+1} = \frac{1}{2+x_n}$.

(a) Prove that (x_n) is contractive.
(b) Find the limit of (x_n).

Question A.5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3 - \frac{x}{2}$. Define a sequence by $x_1 = 10$ and $x_{n+1} = f(x_n)$.

(a) Prove that x_n converges.
(b) Prove that $x = \lim_{n\to\infty} x_n$ is the unique real number satisfying $f(x) = x$.
(c) Prove that the value of the limit x does not depend on the initial choice of x_1.

B. Submitted Questions

Question B.1. Suppose (x_n) is a bounded, increasing sequence. Prove directly (without referring to the Monotone Convergence Theorem or the Cauchy Convergence Criterion) that (x_n) is Cauchy.

Question B.2. Let $x_1 = 4$ and $x_{n+1} = 2 + \frac{1}{x_n}$. Prove that (x_n) converges and find the limit.

C. Challenge Questions

Question C.1. *Contraction Mapping Theorem.* Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is *contractive*. That is, there exists $0 \le \ell < 1$ such that $|f(x) - f(y)| \le \ell |x - y|$ for all $x, y \in \mathbb{R}$. Let $x_1 \in \mathbb{R}$ and define a recursive sequence by $x_{n+1} = f(x_n)$. You may assume $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.

(a) Prove that there exists $x \in \mathbb{R}$ such that $x_n \to x$.
(b) Prove that x must satisfy $f(x) = x$, and is the unique solution to this equation.
(c) Deduce that x is independent of the initial choice of x_1.

Question C.2. *Completion of* $\mathbb{Q}$. Consider the set $\mathcal{C}(\mathbb{Q})$ of Cauchy sequences with all terms in $\mathbb{Q}$. Define a relation $\sim$ on $\mathcal{C}(\mathbb{Q})$ by $(x_n) \sim (y_n)$ if and only if $(x_n - y_n) \to 0$.

(a) Show that $\sim$ is an equivalence relation on $\mathcal{C}(\mathbb{Q})$.
(b) Let X^* be the set of equivalence classes of $\sim$. Let $X, Y \in X^*$ and for $(x_n) \in X$, $y_n \in Y$, define $D\colon x^* \times X^* \to \mathbb{R}$ by $D(X, y_n) = \lim_{n\to\infty} |x_n - y_n|$. Prove that D is well-defined; that is, it does not depend on the chosen representatives x_n and y_n.
(c) Prove the following.
  (i) $D(X, Y) = 0$ if and only if $X = Y$.
  (ii) $D(X, Y) = D(Y, X)$ for all $X, Y \in X^*$.
  (iii) $D(X, Y) \le D(X, Z) + D(Z, Y)$ for all $X, Y, Z \in X^*$.
(d) (*Hard*) Prove that X^* is complete; that is, all Cauchy sequences converge in X^*. (Cauchy and convergence are defined analogously to in the real numbers, but with D instead of the absolute value used to measure distance).