MTH435 - HOMEWORK 3

Solutions to the questions in Section B should be submitted by the start of class on 10/10/18.

A. Warm-up Questions

Question A.1. Exercises from class.

(a) Show that the following are equivalent for a sequence (x_n).
  (i) (x_n) does not converge to x.
  (ii) There exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $n_k \geq k$ such that $|x_{n_k} - x| \geq \varepsilon$.
  (iii) There exists a subsequence (x_{n_k}) such that $|x_{n_k} - x| \geq \varepsilon$ for all $k \in \mathbb{N}$.

(b) Let (x_n) be a bounded sequence and as in class, define $u_n = \sup\{x_m \mid m \geq n\}$ and $v_n = \inf\{x_m \mid m \geq n\}$.
  (i) Prove that (u_n) is decreasing and (v_n) is increasing.
  (ii) Prove that (u_n) and (v_n) converge.

Question A.2. Give examples of the following.

(a) A sequence with infinitely many ceiling terms[1] and infinitely many floor terms.
(b) A sequence with no floor terms and no ceiling terms.
(c) An unbounded sequence which has a convergent subsequence.

Question A.3. For the following recursive sequences, decide if $x = \lim_{n\to\infty} x_n$ exists and if possible, find x.

(a) $x_1 = 100$ and $x_{n+1} = \frac{x_n}{10} + 9$.
(b) $x_1 = 2$ and $x_{n+1} = x_n^2 - 1$.
(c) $x_1 = \frac{5}{2}$ and $x_{n+1} = \frac{1}{5}(x_n^2 + 6)$.
(d) $x_1 = 4$ and $x_{n+1} = 2 + \frac{1}{x_n}$.
(e) $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n}$.

Question A.4. Compute $\limsup x_n$ and $\liminf x_n$ for the following sequences.

(a) $x_n = \sin\left(\frac{n\pi}{2}\right)$.
(b) $x_n = 1 + \frac{\sin(n)}{n}$.
(c) $x_n = (-1)^n\left(1 + \frac{1}{n}\right)$.

Question A.5. For the following pairs of sequences (x_n) and (y_n), compute $\liminf(x_n) + \liminf(y_n)$, $\liminf(x_n + y_n)$ and $\limsup(x_n) + \liminf(y_n)$.

(a) $x_n = 1 + \frac{1}{n}$ and $y_n = 1 - \frac{1}{n}$.
(b) $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.
(c) $x_n = y_n = (-1)^n$.

Question A.6. Let x_n be a bounded sequence. Prove that

$$\liminf(x_n) = -\limsup(-x_n).$$

B. Submitted Questions

Question B.1. Consider the recursive sequence defined by $x_1 = 11$ and $x_{n+1} = \sqrt{5 + x_n}$. Prove that (x_n) converges and find the limit.

Question B.2. Suppose (x_n) and (y_n) are bounded sequences. Prove that

$$\liminf(x_n) + \liminf(y_n) \leq \liminf(x_n + y_n) \leq \limsup(x_n) + \liminf(y_n).$$

C. Challenge Questions

Question C.1. Let $a, b > 0$. Let (x_n) be defined recursively by $x_1 = a$ and $x_{n+1} = \sqrt{b + x_n}$. Show that (x_n) converges to $\frac{1+\sqrt{1+4b}}{2}$. *Hint: To show monotonicity, it may help to prove the equality* $(x_{n+1} - x_n)(x_{n+1} + x_n) = x_n - x_{n-1}$ *and compare signs.*

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[1] Analogously to floor terms, a term x_C is a ceiling term if $x_C \geq x_n$ for all $n \geq C$.