MTH435 - HOMEWORK 3

Solutions to the questions in Section B should be submitted by the start of class on 10/03/18.

A. Warm-up Questions

Question A.1. Exercises from class.

(i) Suppose $x_n \to x$ and $c \in \mathbb{R}$. Show that $\lim_{n\to\infty} cx_n = cx$.
(ii) Suppose $x_n \to x > 0$. Show that eventually $x_n \geq 0$.

Question A.2. Let $x, y \in \mathbb{R}$. Prove the following identities.

(i) $\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$.
(ii) $\min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$.

Now suppose (x_n) and (y_n) are convergent sequences and define the sequences $M_n = \max\{x_n, y_n\}$ and $m_n = \min\{x_n, y_n\}$. Prove that (M_n) and (m_n) converge.

Question A.3. Use the limit laws to find the limits of the following sequences.

(i) $\frac{\sin(\sqrt{n})+5}{n}$.
(ii) $\frac{3n^2+5n+19}{4n^2+16}$.
(iii) $\frac{3+(-1)^n n}{n^2}$.
(iv) $\frac{2^{3n}}{3^{2n}}$.
(v) $\sqrt{n^2 + 5n} - n$.
(vi) $\sqrt[n]{a^n + b^n}$ where $0 < a < b$ (Hint: Find good *b*read for your sandwich!)

Question A.4. Prove or disprove the following for sequences (x_n) and (y_n).

(i) If $(x_n + y_n)$ converges then (x_n) and (y_n) converge.
(ii) If $(x_n y_n)$ converges and (x_n) converges then (y_n) converges.
(iii) If $(x_n y_n)$ converges and (x_n) converges to $x \neq 0$ then (y_n) converges.
(iv) If (x_n) converges and $y_n = x_{n+1} - x_n$ then (y_n) converges.
(v) If $y_n = x_{n+1} - x_n$ converges to 0 then (x_n) converges.

B. Submitted Questions

Question B.1. Suppose that for all $\varepsilon > 0$ there exists K such that $n \geq K$ implies $|x_n - y_n| < \varepsilon$. Prove that (x_n) converges if and only if (y_n) converges.

Question B.2. Suppose $x_n \to 0$ and (y_n) is bounded. Prove or disprove that $(x_n y_n)$ is convergent.

C. Challenge Questions

Question C.1. You may find previous homework sheets useful for this. Let $\mathcal{T} = \{U \subseteq \mathbb{R} \mid U \text{ is open}\}$.

(T1) Show that $\mathbb{R}, \varnothing \in \mathcal{T}$.
(T2) Show that if $(U_t)_{t\in T}$ is any family of members of $\mathcal{T}$, then $\bigcup_{t\in T} U_t$ is in $\mathcal{T}$.
(T3) Show that if $U_1, U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.

Any collection of subsets $\mathcal{T}$ satisfying (T1), (T2) and (T3) is called a *topology* for $\mathbb{R}$.

Question C.2. Prove the following are topologies on $\mathbb{R}$.

(i) $\mathcal{T} = \mathcal{P}(\mathbb{R}) = \{U \mid U \subseteq \mathbb{R}\}$.
(ii) $\mathcal{T} = \{\varnothing, \mathbb{R}\}$.
(iii) $\mathcal{T} = \{U \subseteq \mathbb{R} \mid \mathbb{R} - U \text{ is finite}\} \cup \{\varnothing\}$.