MTH435 - HOMEWORK 2

Solutions to the questions in Section B should be submitted by the start of class on 09/26/18.

A. Warm-up Questions

Question A.1. We investigate the necessity of the assumptions in the Nested Intervals Property.

(i) *Closed*: Let $I_n = \left(0, \frac{1}{n}\right)$. Prove that $\bigcap I_n = \varnothing$.
(ii) *Bounded*: Let $I_n = [n, \infty)$. Prove that $\bigcap I_n = \varnothing$.
(iii) *Nested*: Let $I_n = [n, n+1]$. Prove that $\bigcap I_n = \varnothing$.
(iv) *Completeness*: Let $I_n = \left[\sqrt{2}, \sqrt{2} + \frac{1}{n}\right] \cap \mathbb{Q}$. Prove that $\bigcap I_n = \varnothing$.

Question A.2. Use the definition of convergence to prove the following limits.

(i) $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$
(ii) $\lim_{n\to\infty} \dfrac{1}{n}(\cos(n^2+5) - \sin(2n^3)) = 0$
(iii) $\lim_{n\to\infty} \sqrt{4 + \dfrac{1}{n}} = 2$
(iv) (*Harder?*) $\lim_{n\to\infty} \sqrt{n(n+1)} - n = \dfrac{1}{2}$

Question A.3. Prove or provide a counterexample to the following.

(i) If $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ then $x \geq 0$.
(ii) If $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ then $x > 0$.

Question A.4. Prove that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $x_n \to x$ then $\sqrt{x_n} \to \sqrt{x}$. You may like to use the following steps.

(i) Show that if $x_n \to 0$ then $\sqrt{x_n} \to 0$.
(ii) Prove that if $a, b \geq 0$ then
$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}.$$
(iii) Use part (ii) to show that if $x > 0$ and $x_n \to x$ then $\sqrt{x_n} \to \sqrt{x}$.

B. Submitted Questions

Question B.1. Suppose $I_n = [a_n, b_n]$ is a nested sequence of closed, bounded intervals. Define $\xi = \sup\{a_n \mid n \in \mathbb{N}\}$ and $\eta = \inf\{b_n \mid n \in N\}$. Prove that $\bigcap_{n\in\mathbb{N}} I_n = [\xi, \eta]$.

Question B.2. Prove or give a counterexample to the following. Let (x_n) be a sequence.

(i) If (x_n) converges then $(|x_n|)$ converges.
(ii) If $(|x_n|)$ converges then x_n converges.

C. Challenge Questions

Question C.1. Show that $x_n \to x$ if and only if for every neighbourhood U of x there exist only finitely many $n \in \mathbb{N}$ such that $x_n \notin U$.

Question C.2. Let $A \subset \mathbb{R}$. We call $x \in \mathbb{R}$ a *limit point* of A if for every neighbourhood U of x there exists $a \in A$ such that $a \in U$. Denote the set of limit points of A by $\overline{A}$[1].

(i) Compute $\overline{(0,1)}, \overline{[0,1]}, \overline{\{1/n \mid n \in \mathbb{N}\}}$ and $\overline{\mathbb{Q}}$.
(ii) Show that $x \in \overline{A}$ if and only if there exists a sequence $(x_n) \to x$ with $x_n \in A$ for all $n \in \mathbb{N}$.
(iii) Prove that A is closed if and only if $A = \overline{A}$.

Question C.3. Let $a > 0$ and $n \in \mathbb{N}$. By following the argument that $\sqrt{2}$ exists, or otherwise, prove that there exists $b \in \mathbb{R}$ such that $b^n = a$.

[1] This set is called the *closure* of A