MTH435 - HOMEWORK 10

Here are some questions (mostly on the EVT and IVT) to help you prepare for the final.

Question 1. Let $f\colon [0,1] \to [0,1]$ be continuous. Prove that there exists $c \in [0,1]$ such that $f(c) = c$. Is this true if we replace $[0,1]$ by $(0,1)$?

Question 2. Prove that if f is a polynomial of odd degree, then f has a real root.

Question 3. Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous and $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all x.

Question 4. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at c and that f takes both positive and negative values in every neighbourhood of c. Prove that $f(c) = 0$.

Question 5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous with $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = 0$. Show that f has a global maximum or a global minimum on $\mathbb{R}$, but may not have both.

Question 6. Let $f\colon [0,1] \to [0,1]$ be continuous with $f(0) = f(1)$. Prove that there exists $c \in \left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.

Question 7. Let $f\colon [a,b] \to \mathbb{R}$ satisfy $f(x) > 0$ for all $x \in [a,b]$. Prove there exists $y > 0$ such that $f(x) \geq y$ for all $x \in [a,b]$.

Question 8. Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous and define $E = \{x \in [a,b] \mid f(x) = g(x)\}$. Prove that if (x_n) is a sequence with $x_n \in E$ for all n and $x_n \to x$, then $x \in E$.