# MTH435 - HOMEWORK 1

Solutions to the questions in Section B should be submitted by the start of class on 09/19/18.

## A. Warm-up Questions

**Question A.1.** Solve the following inequalities.

(i) $x < \frac{1}{x}$.

(ii) $|x+1| + |x-2| \leq 2$.

(iii) $x - 1 \leq |x-2|$.

(iv) $|x^2 - 2| < 3$.

**Question A.2.** Let $a, b \in \mathbb{R}$. Prove the following.

(i) $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$.

(ii) $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$.

(iii) $\max\{\max\{a, b\}, c\}$.

**Question A.3.** Prove the following for subsets of $\mathbb{R}$.

(i) $\mathbb{R}$ is open and closed.

(ii) $\varnothing$ is open and closed.

(iii) $[0, 1)$ is neither open nor closed.

(iv) $\{x_1, x_2, \ldots, x_n\}$ is closed.

(v) $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ is neither open nor closed.

**Question A.4.** Let $U_1$ and $U_2$ be open sets in $\mathbb{R}$. Prove the following.

(i) $U_1 \cup U_2$ is open.

(ii) $U_1 \cap U_2$ is open.

**Question A.5.** Compute $\sup X$ and $\inf X$ (if they exist) for the following sets.

(i) $(0, 1) \cup (2, 3]$.

(ii) $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$

(iii) $\{x \in R \mid x < \frac{1}{x}\}$

(iv) $\{x \in \mathbb{R} \mid \sin \frac{1}{x} = 0\}$.

**Question A.6.** Let $S$ and $T$ be non-empty bounded subsets of $\mathbb{R}$. Define $-S = \{-s \mid s \in S\}$ and $S + T = \{s + t \mid s \in S, t \in T\}$. Prove the following.

(i) $\sup(-X) = -\inf X$.

(ii) $\sup(S + T) \leq \sup S + \sup T$.

## B. Submitted Questions

**Question B.1.** Let $X = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$. Is $X$ open, closed, both or neither? Justify your answer.

**Question B.2.** Let $\varnothing \neq S \subseteq T \subseteq \mathbb{R}$ and suppose $T$ is bounded. Prove that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

## C. Challenge Questions

**Question C.1.** Using the properties of $\mathbb{R}$ given on pages 24 and 26, prove the following. (This exercise is not really important for analysis but it is good for the soul). Let $a, b, c, d \in \mathbb{R}$.

(i) If $a + b = 0$ then $b = -a$.

(ii) $-(-a) = a$.

(iii) If $a^2 = a$ then $a \in \{0, 1\}$.

(iv) If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.

**Question C.2.** Let $(U_\alpha)_{\alpha \in A}$ be an indexed family of open sets. Prove or disprove the following.

(i) $\bigcup_{\alpha \in A} U_\alpha$ is open.

(ii) $\bigcap_{\alpha \in A} U_\alpha$ is open.