# MTH307 - HOMEWORK 8

Solutions to the questions in Section B should be submitted by the start of class on 4/12/18.

## A. Warm-up Questions

We use the follwing terminology.

$$\mathsf{P} = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \mid a_N, a_{n-1}, \ldots, a_0 \in \mathbb{R}, n \geq 0\}$$

is the set of all real polynomials. If $A \subseteq \mathbb{R}$, we say that $f\colon A \to \mathbb{R}$ is *increasing* if $x < y$ implies $f(x) < f(y)$.

**Question A.1.** Compute the domain and range of the following functions and justify your answer.

(i) $f(x) = \frac{x}{x+1}$.
(ii) $f(x) = x^2 - 4x - 7$.
(iii) $f(x) = \frac{2x-1}{x+5}$.
(iv) $f(x, y) = \frac{x+y}{x-y}$.
(v) $f(x) = \frac{ax+b}{cx+d}$ for $a, b, c, d \in \mathbb{R}$.

**Question A.2.** Find $f \circ g$ and $g \circ f$ for the following pairs of functions.

(i) $f(x) = \frac{x}{x^2+1}$ and $g(x) = 2x - 1$.
(ii) $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = g(x, y) = (xy, x^3)$.
(iii) $f, g\colon \mathbb{R} - \{-1\} \to \mathbb{R}$, $f(x) = g(x) = \frac{1}{x+1} - 1$.

**Question A.3.** For each of the following functions, decide if they are injective, surjective or bijective and prove your answer.

(i) $f\colon \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$, $f(x) = \left(\frac{x+1}{x-1}\right)^3$.
(ii) $f\colon \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$, $f(X) = \overline{X}$.
(iii) $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(m, n) = 2^m 3^n$.
(iv) $f\colon \mathbb{Z} \to \mathbb{Z}$,

$$f(n) = \begin{cases} n+3 & \text{if } n \text{ is even,} \\ 2(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

(v) $f\colon \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$, $f(x) = \frac{5x+1}{x-2}$.
(vi) $F\colon \mathsf{P} \to \mathsf{P}$, $F(p) = p'$.
(vii) $F\colon \mathsf{P} \to \mathsf{P}$, $F(p) = p + 2p'$.

**Question A.4.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be increasing. Prove that $f$ is injective.

**Question A.5.** Suppose $f\colon X \to Y$ and $g\colon Y \to Z$.

(i) Show that if $f$ and $g$ are injective then $g \circ f$ is injective.
(ii) Show that if $g \circ f$ is injective then $f$ is injective.
(iii) Suppose $g \circ f$ is injective. Must it be true that $g$ is injective? Prove it or provide a counterexample.

## B. Submitted Questions

**Question B.1.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x + y, x^2)$. Decide if $f$ is injective, surjective or bijective and prove your answer.

**Question B.2.** Let $f\colon X \to Y$ and $g\colon Y \to Z$ be surjective. Prove that $g \circ f$ is surjective.

## C. Challenge Questions

**Question C.1.** Let $L\colon [0, 1] \to \mathbb{R}$, $L(x) = 4x(1 - x)$ and $T\colon [0, 1] \to \mathbb{R}$ by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let $h\colon [0, 1] \to [0, 1]$, $h(x) = \sin^2\left(\frac{\pi x}{2}\right)$. Show that $L \circ h = h \circ T$.