MTH307 - HOMEWORK 7

Solutions to the questions in Section B should be submitted by the start of class on 4/5/18.

A. Warm-up Questions

Question A.1. Compute $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ for the following indexed sets. You don't need to prove your answer.

(i) $A_n = \{0, 1, 2, \ldots, n\}$.
(ii) $A_n = \{-n, 0, n\}$.
(iii) $A_n = [n, n+1]$.
(iv) $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$.
(v) $A_n = \left(0, \frac{1}{n}\right)$.
(vi) $A_n = [-1, n]$ if n is even and $A_n = \left[-\frac{1}{n}, 1\right]$ if n is odd.
(vii) $A_n = \{0, (-1)^n n\}$.
(viii) $A_n = \left\{\frac{1+(-1)^n}{n}, \frac{-1-(-1)^n}{n}\right\}$.

Question A.2. Compute

$$\bigcup_{X \in \mathcal{P}(\mathbb{Z})} X \qquad \text{and} \qquad \bigcap_{X \in \mathcal{P}(\mathbb{Z})} X$$

Question A.3. For $x \in (0, 1]$, define $A_x = [0, x]$.

(i) Show that $0 \in A_x$ for all x.
(ii) Show that if $y > 0$, then $y \notin A_x$ for $x = \frac{\sqrt{y}}{2}$.
(iii) Prove that $\bigcap_{x \in (0,1]} A_x = \{0\}$.

Question A.4. For each $\alpha \in \mathbb{R}$, define $X_\alpha = \{(x, \alpha x(x-1)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$.

(i) Sketch the sets X_0, X_1, X_{-1} and X_2.
(ii) Show for all $\alpha \in \mathbb{R}$ that $(0, 0)$ and $(1, 0)$ are in X_α. Deduce that $\{(0, 0), (1, 0)\} \subseteq \bigcap_{\alpha \in \mathbb{R}} X_\alpha$.
(iii) Show that $X_0 \cap X_1 = \{(1, 0), (0, 0)\}$ and deduce that $\bigcap_{\alpha \in \mathbb{R}} X_\alpha \subseteq \{(0, 0), (1, 0)\}$.
(iv) Hence deduce that $\bigcap_{\alpha \in \mathbb{R}} X_\alpha = \{(0, 0), (1, 0)\}$.

B. Submitted Questions

Question B.1. Prove that

$$\bigcap_{x \in \mathbb{R}} [1 - x^2, 2 + x^2] = [1, 2]$$

Question B.2. For each $\alpha \in \mathbb{R}$, define $X_\alpha = \{(x, 2^{\alpha x}) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. Find $\bigcap_{\alpha \in \mathbb{R}} X_\alpha$ and prove your answer.

C. Challenge Questions

Question C.1. For $a, b \in \mathbb{R}^2$, define $P_{a,b} = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by = 0\}$. Compute $\bigcap_{(a,b) \in \mathbb{R}^2} P_{a,b}$ and $\bigcup_{(a,b) \in (\mathbb{R}^2 - \{(0,0)\})} P_{a,b}$.

Question C.2. For $x \in [0, 1]$, define

$$A_x = [x, 1] \times [0, e^x].$$

Compute $\bigcup_{x \in [0,1]} A_x$ and $\bigcap_{x \in [0,1]} A_x$.