MTH307 - HOMEWORK 5

Solutions to the questions in Section B should be submitted by the start of class on 3/22/18.

A. Warm-up Questions

Question A.1. Compute the following sets.

(i) $\mathcal{P}(\{a,b\}) \times \mathcal{P}(\{1,2\})$.
(ii) $\mathcal{P}(\{a,b\} \times \{1,2\})$.
(iii) $\mathcal{P}(\mathcal{P}\{\varnothing, \{\varnothing\}\})$.
(iv) $\mathcal{P}(\mathbb{N} \cap \mathcal{P}(\mathbb{N}))$.
(v) $\{\varnothing\} \cap \mathcal{P}(\{1,2,3\})$.
(vi) $\mathcal{P}(\{\mathbb{Z}, \mathbb{Q}\})$.

Question A.2. Let $A = \{2,4,6\}$, $B = \{1,3,5\}$ and $C = \{2,3,5\}$ with universal set $\mathcal{U} = \{1,2,3,4,5,6\}$. Compute the following.

(i) $\mathcal{P}(A) - \mathcal{P}(C)$.
(ii) $\mathcal{P}(A - C)$.
(iii) $\overline{A} \cap C$.
(iv) $(A - C) \cup (C - A)$.
(v) $\overline{B} - A$.
(vi) $A - (B \cup C)$.
(vii) $(A - B) \cup C$.
(viii) $\mathcal{P}(A \cap C) \times B$.

Question A.3. Let A, B, C and D be subsets of some universal set $\mathcal{U}$. Prove or disprove the following.

(i) $\overline{\overline{A}} = A$.
(ii) $A \cap \overline{A} = \varnothing$.
(iii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
(iv) $A \times \varnothing = \varnothing$.
(v) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(vi) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.
(vii) If $A \times B = A \times C$ then $B = C$.
(viii) $A \subseteq B$ if and only if $A - B = \varnothing$.
(ix) If $A \neq \varnothing$ then $A \times B \subseteq A \times C$ if and only if $B \subseteq C$.
(x) If $B \neq \varnothing$ and $A \times B \subseteq B \times C$ then $A \subseteq C$.

B. Submitted Questions

Question B.1. Prove or disprove the following for arbitrary sets A, B and C.

(i) $A \times B = B \times A$ if and only if $A = B$.
(ii) $A \times (B - C) = (A \times B) - (A \times C)$.

C. Challenge Questions

Question C.1. It is possible to define ordered pairs by using sets. A definition due to Kuratowski defines

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Prove that this definition satisfies the condition that $(x, y) = (u, v)$ if and only if $x = u$ and $y = v$. Can you suggest a set-theoretic definition for (x, y, z)?

Question C.2. Let X be a set. For $A, B \in \mathcal{P}(X)$ define the symmetric difference $A \triangle B = (A - B) \cup (B - A)$. Prove the following for all $A, B, C \in \mathcal{P}(X)$.

(i) $A \triangle B = (A \cup B) - (A \cap B)$
(ii) $A \triangle B = B \triangle A$.
(iii) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.
(iv) There exists a unique $E \in \mathcal{P}(X)$ such that $A \triangle E = A = E \triangle A$ for all $A \in \mathcal{P}(X)$.
(v) For E in the previous part, show that for each $A \in \mathcal{P}(X)$ there exists a unique $B \in \mathcal{P}(X)$ such that $A \triangle B = E = B \triangle A$.

For those taking MTH316 in the future, this exercise shows that $(\mathcal{P}(X), \triangle)$ forms a (commutative) group.