MTH307 - HOMEWORK 5

Solutions to the questions in Section B should be submitted by the start of class on 3/22/18.

A. Warm-up Questions

Question A.1. Let F_n be the nth Fibonacci number. Prove the following.

(i) For each $n \in \mathbb{N}$, we have $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$.
(ii) For all $n \in \mathbb{N}$ we have $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$.
(iii) If $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = \frac{1}{3}(2a_n + a_{n+1})$ for all $n \in \mathbb{N}$ then $1 \leq a_n \leq 2$ for all n.
(iv) If $a_1 = 2$, $a_2 = 5$ and $a_{n+2} = 2a_n - a_{n+1}$, find a closed formula for a_n and prove it by induction. (Hint: compare your sequence to the values of 2^{n-1}.)

Question A.2. Let $A = \{1, \{1,2\}, \{2,3\}, 3\}$ and $B = \{\varnothing, \{\varnothing\}\}$. Decide if the following are true.

(i) $\{1.2\} \in A$.
(ii) $\{1,2\} \subseteq A$.
(iii) $\{1,3\} \subseteq A$.
(iv) $\varnothing \in B$.
(v) $\{\varnothing\} \in B$.
(vi) $\{\varnothing\} \subseteq B$.
(vii) $\varnothing \in \varnothing$.

Question A.3. Let A and B be sets. Prove the following.

(i) $A \subseteq A$.
(ii) $A \subseteq A \cup B$.
(iii) $A \cap B \subseteq A$.
(iv) $\varnothing \subseteq A$.
(v) $A \cup A = A$.
(vi) $A \cap A = A$.
(vii) $A \cup B = B \cup A$.
(viii) $A \cap B = B \cap A$.

Question A.4. Prove or disprove the following.

(i) $\{n \in \mathbb{Z} \mid 6|n\} \subseteq \{x \in \mathbb{Z} \mid 2|x\}$.
(ii) $\{2n \mid n \in \mathbb{N}\} \cap \{3m \mid m \in \mathbb{Z}\} \neq \varnothing$.
(iii) $\{9^n \mid n \in \mathbb{Z}\} \subseteq \{3^n \mid n \in \mathbb{Z}\}$
(iv) $\{9^n \mid n \in \mathbb{Q}\} = \{3^n \mid n \in \mathbb{Q}\}$.

Question A.5. Let A, B, C and D be sets. Prove or disprove the following.

(i) $A \cup B = A \cup (B - A)$.
(ii) If $A = B - C$ then $B = A \cup C$.
(iii) If $A \subseteq B \cap C$ then $A \subseteq B$ and $A \subseteq C$.
(iv) $A \cup (B \cup C) = (A \cup B) \cup C$.
(v) If $A \subseteq B \cup C$ and $A \cap B = \varnothing$ then $A \subseteq C$.
(vi) If $A \subseteq C$ and $B \subseteq D$ then $B - C \subseteq D - A$.
(vii) If $B = A \cup C$ then $A = B - C$.
(viii) $A - C \subseteq B - C$ if and only if $A \subseteq B$.

B. Submitted Questions

Question B.1. Prove or disprove the following.

(i) If A, B and C are sets then $(A \cup B) - C = (A - C) \cup (B - C)$.
(ii) If A, B and C are sets and $A \subseteq B \cup C$ then $A \subseteq B$ or $A \subseteq C$.

C. Challenge Questions

Question C.1. Prove the following.

(i) $\{2n \mid n \in \mathbb{Z}\} \cap \{3n \mid n \in \mathbb{Z}\} \cap \{5n \mid n \in \mathbb{Z}\} = \{30n \mid n \in \mathbb{Z}\}$.
(ii) If A and B are sets then $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Question C.2. *A Preview of MTH435.* We say that $X \subseteq \mathbb{R}$ is open if for all $x \in X$ there exists $r > 0$ such that the interval $\{y \in \mathbb{R} \mid x - r < y < x + r\} = (x - r, x + r) \subseteq X$. We say a set Y is closed if $\mathbb{R} - Y$ is open.

(i) Prove that $\mathbb{R}$ is open and closed.
(ii) Prove that $\varnothing$ is open and closed.
(iii) Prove that $(0,1)$ is open.
(iv) Prove that $\{0\}$ is closed.