MTH307 - HOMEWORK 4

Solutions to the questions in Section B should be submitted by the start of class on 3/1/18.

A. Warm-up Questions

Question A.1. Prove the following by contradiction.

(i) There is no integer n which is both odd and even.
(ii) There are no integers m and n such that $21m + 30n = 1$.
(iii) $\sqrt[3]{2}$ is irrational.
(iv) If $\pi/2 \leq x \leq \pi$ then $\sin x - \cos x \geq 1$.

Question A.2. Prove the following with the contrapositive.

(i) Let a, b and c be integers. If a does not divide bc, then a does not divide b.
(ii) Let $x \in \mathbb{Z}$. If $x^3 - 1$ is even then x is odd.
(iii) Let $x, y \in \mathbb{Z}$. Then if $x + y$ is even then x and y have the same parity.
(iv) Let $x, y \in \mathbb{Z}$. If $x^2(y^2 - 2y)$ is odd then x and y are odd.

Question A.3. Prove or disprove the following.

(i) Let a and b be positive integers. Then if $a \mid b$ and $b \mid a$ then $a = b$.
(ii) If x and y are irrational then $x + y$ is irrational.
(iii) If $x \neq 0$ and x is rational and y is irrational then xy is irrational.
(iv) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})[xy = xz]$.
(v) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(\exists z \in \mathbb{R})[xy = xz]$.
(vi) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[xy = 1]$.
(vii) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})[xy = 1]$.
(viii) $(\forall x \in \mathbb{Q})(\exists y \in \mathbb{Z})[xy \in \mathbb{Z}]$.

Question A.4. Prove the following by induction.

(i) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
(ii) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.
(iii) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n}(2i - 1) = n^2$.
(iv) For all $n \geq 5$ we have $n^2 < 2^n$.
(v) For all $n \in \mathbb{N}$, we have $8 \mid (3^{2n} - 1)$.
(vi) For all $n \in \mathbb{N}$, we have $3 \mid (n^2 + 5n + 6)$.

B. Submitted Questions

Question B.1. Prove or disprove the following.

(i) If $x, y \in \mathbb{Z}$ then $x^2 - 4y \neq 2$.
(ii) Let $a, b \in \mathbb{R}$. Then if $a^3 + ab^2 \leq b^3 + a^2b$ then $a \leq b$.

C. Challenge Questions

Question C.1. Let $x \in \mathbb{R}$. Prove that $\sqrt{3} + x$ is irrational or $\sqrt{3} - x$ is irrational.

Question C.2. Let $x \in \mathbb{R}$ and suppose that for all $\varepsilon > 0$ we have $|x| < \varepsilon$. Prove that $x = 0$.

Question C.3. Let

$$f(x) = \begin{cases} 7 - 3x & \text{if } x \neq 1, \\ 2018 & \text{if } x = 1. \end{cases}$$

Prove that $\lim_{x \to 1} f(x) = 4$.

Question C.4. Prove that if $x > -1$ and $n \in \mathbb{N}$ then $(1 + x)^n \geq 1 + nx$.

Question C.5. Prove for all integers $n \in \mathbb{N}$ that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$