Matings of Cubic Polynomials with a Fixed Critical Point

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21st May 2014
Iberoamerican Congress 2014, CUNY

Outline

Mating for Beginners

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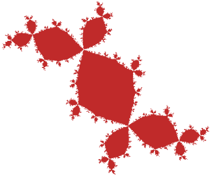

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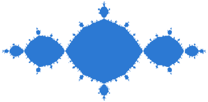

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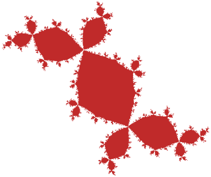
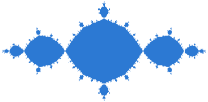


$\perp\!\!\!\perp$



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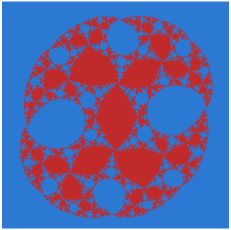

Topological Matings

Let f_i $(i = 1, 2)$ be monic degree d polynomials with filled Julia set $K_i = K(f_i)$ and associated Carathéodory loop γ_i. We define the topological mating $f_1 \perp\!\!\!\perp f_2$ on a topological space $K_1 \perp\!\!\!\perp K_2$:

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Topological Mating

- Take the disjoint union of K_1 and K_2.
- $K_1 \perp\!\!\!\perp K_2$ is the quotient space formed by identifying $\gamma_1(\theta)$ with $\gamma_2(-\theta)$.
- The maps f_i on the K_i fit together to form a new map $f_1 \perp\!\!\!\perp f_2$.

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We say f_1 and f_2 are topologically mateable if the quotient space $K_1 \perp\!\!\!\perp K_2$ is a sphere. We say they are (geometrically) mateable if in addition $f_1 \perp\!\!\!\perp f_2$ is Thurston equivalent to a rational map on $\widehat{\mathbb{C}}$.

Why matings?

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But when does mating two polynomials f_1 and f_2 with filled Julia sets K_1 and K_2 produce a rational map?

- When is the quotient space $K_1 \perp\!\!\!\perp K_2$ Hausdorff?
- When is the quotient space $K_1 \perp\!\!\!\perp K_2$ a sphere?
- When is the resulting dynamics equivalent to a rational map? (i.e is there a Thurston obstruction?)

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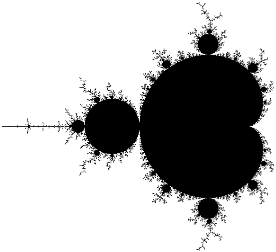
Quadratic Case

The quadratic (or bicritical) case is reasonably well understood:



Theorem (Rees, Shishikura, Tan)

In the bicritical case, if f_1 and f_2 do not lie in conjugate limbs of $\mathcal{M}$, then $K_1 \perp\!\!\!\perp K_2$ is homeomorphic to S^2 and we can give this sphere a unique conformal structure to make $f_1 \perp\!\!\!\perp f_2$ a holomorphic degree d rational map.

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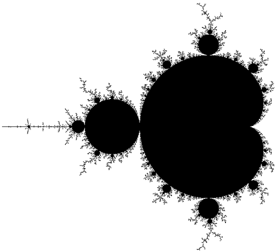

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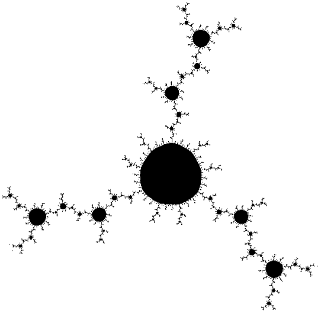
Essentially, this says that in the quadratic case, the quotient is a sphere if and only if the resulting map is equivalent to a rational map. All obstructions are Levy cycles.

Other obstructions

However, there exist other obstructions in higher degrees: Consider the following



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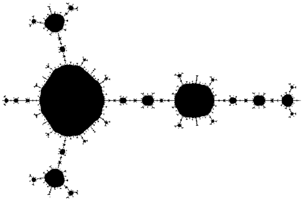

Both polynomials are in $\mathcal{S}_3$. The quotient is a sphere, but the mating is not a rational map.

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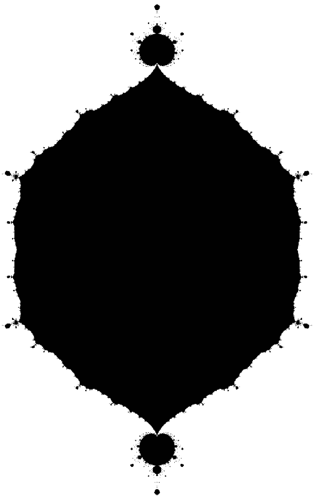
The space $\mathcal{S}_1$.



As mentioned in Bonifant's talk, there is only one escape region in $\mathcal{S}_1$. In particular this means that the intersection of $\mathcal{S}_1$ with the connectedness locus is combinatorially a tree.

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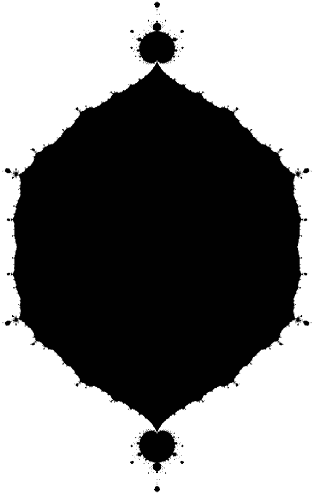

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This is not true for $\mathcal{S}_2, \mathcal{S}_3 \ldots$

Combinatorics and rays

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- If the parameter ray of angle θ lands at a map f in parameter space, then the dynamical ray of angle θ for f lands at the co-critical point.
- If the map f lies in the wake of the rays of angle θ_1 and θ_2, then the dynamical rays of angle θ_1 and θ_2 for f land together on the Julia set of f.

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Let f_1, f_2 be in $\mathcal{S}_1$ with filled Julia sets .

Question

When is the quotient space $K_1 \perp\!\!\!\perp K_2$ a sphere (when are f_1 and f_2 topologically mateable)?

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In other words, when do the ray equivalence classes contain loops?

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In other words, when do the ray equivalence classes contain loops?

Recall that for quadratics, the ray classes contained loops precisely when f_1 and f_2 belonged to conjugate limbs of $\mathcal{M}$.

First the unsurprising news...

Proposition

Let f_1 and f_2 be in conjugate limbs of $\mathcal{S}_1$. Then $K_1 \perp\!\!\!\perp K_2$ is not a sphere.

Partial results

First the unsurprising news…

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Let f_1 and f_2 be in conjugate limbs of $\mathcal{S}_1$. Then $K_1 \perp\!\!\!\perp K_2$ is not a sphere.

However…

Proposition

There exist polynomials f_1 and f_2 where f_1 and f_2 do not lie in conjugate limbs of $\mathcal{S}_1$ but for which the space $K_1 \perp\!\!\!\perp K_2$ is not a sphere.

Example

Let f_1 be the period 2 map in the $\left(\frac{1}{24}, \frac{2}{24}\right)$-limb.

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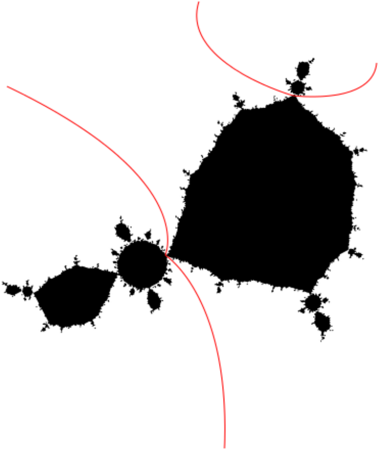

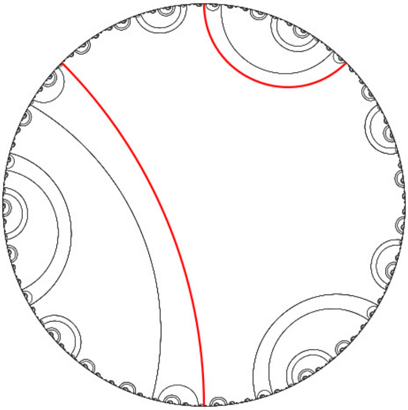

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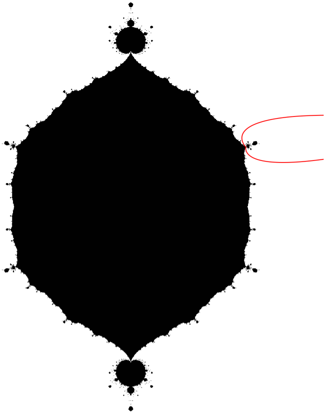
This map resides here in parameter space. . .



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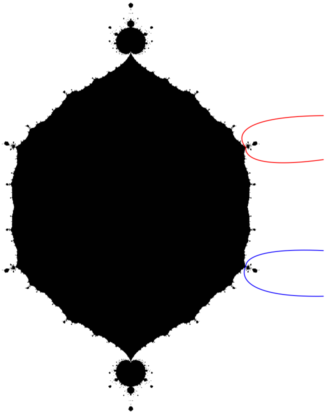
This map resides here in parameter space. . .
. . . and the mating with the conjugate map is obstructed. . .



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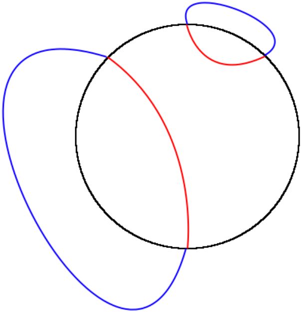

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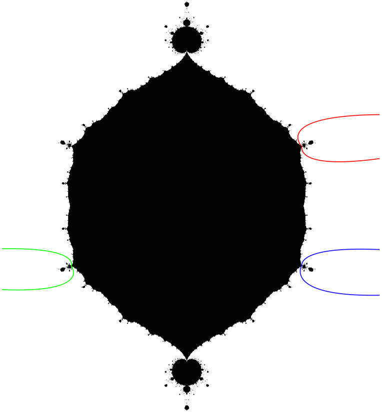
. . . as is the mating with this complementary map.



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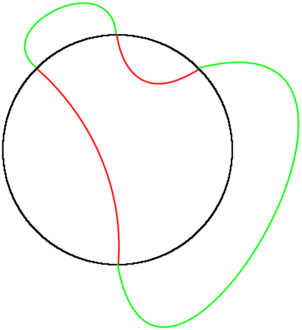

A characterisation of topological obstructions

Clearly, all limbs have a corresponding conjugate limb. But which limbs have a complementary limb?

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Conjecture

Let $\mathcal{C}_t \subset \mathcal{S}_1$ be a limb. Then $\mathcal{C}_t$ has a complementary limb if and only if t is periodic under the map $t \mapsto 2t$ on $\mathbb{R}/\mathbb{Z}$.

Here t represents the internal angle of the limb with respect to the type A component.

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Here t represents the internal angle of the limb with respect to the type A component.

Conjecture

The mating $f_1 \perp\!\!\!\perp f_2$ is topologically obstructed if and only if one of the following occurs.

- f_1 and f_2 lie in conjugate limbs.
- f_1 and f_2 lie in complementary limbs.

Conjecture

The only conformal obstructions to matings are the topological obstructions. In other words, all Thurston obstructions are Levy cycles.

Idea of Proof:

- If a fixed critical point c is contained in a connected component A of $S^2 \setminus \Gamma$:

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- If a fixed critical point c is contained in a connected component A of $S^2 \setminus \Gamma$:
  - If A is a disk then A contains another postcritical point p, and then $\mathrm{orb}(p) \subset A$.
  - If A is not a disk then the only postcritical points in A are the two fixed critical points.

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- In the first case, the two polynomials must be in complementary limbs.
- In the second case, the two polynomials must be in conjugate limbs.

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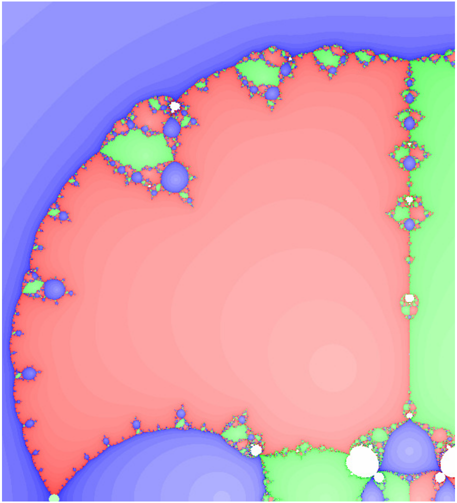
Evidence

Here is an excuse to show a nice picture. . .



Here is a slice of cubic rational maps with two fixed critical points and a period two critical orbit. The only obstructions are the topological ones. Numerical experiments have shown the conjecture to be true for low period polynomials.

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Thank you for listening!