ON THE DECK GROUPS OF ITERATES OF BICRITICAL RATIONAL MAPS

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Abstract. Given a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere, we define $\operatorname{Deck}(f)$ to be the group of Möbius transformations μ satisfying $f \circ \mu = f$. In this note, we consider the groups $\operatorname{Deck}(f^k)$, where f is a *bicritical* rational map (that is, a rational map with exactly two critical points) and f^k denotes the kth iterate of f. In particular, we give a complete description of which groups (up to isomorphism) arise as the groups $\operatorname{Deck}(f^k)$ for bicritical rational maps f.

1. Introduction

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map on the Riemann sphere. We will denote the set of critical points (resp. critical values) of f by $\mathcal{C}_f$ (resp. $\mathcal{V}_f$). A rational map f is called bicritical if $|\mathcal{C}_f| = 2$. In addition, a bicritical rational map f is called a power map if $\mathcal{C}_f = \mathcal{V}_f$. This is equivalent to the condition that f is conjugate to $z \mapsto z^{\pm d}$ for some $d \geq 2$.

In this note, we study particular groups of symmetries of the iterates of biciritical rational maps.

Definition 1.1. *The* deck group *of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the group*

$$\operatorname{Deck}(f) \coloneqq \{\mu \in \text{Möb} \mid f \circ \mu = f\},$$

where Möb denotes the group of Möbius transformations of the Riemann sphere $\hat{\mathbb{C}}$.

Deck groups were used in [4] as a tool for characterizing bicritical rational maps with shared iterates. Given a rational map f, the deck group $\operatorname{Deck}(f)$ is a finite subgroup of Möb (in fact, $|\operatorname{Deck}(f)| \leq \deg(f)$). It is well-known that a finite group of Möbius transformations is isomorphic to either a cyclic group $\mathbb{Z}_n$ ($n \in \mathbb{N}$), a dihedral group D_n ($n \in 2\mathbb{N}$), or one of the *polyhedral groups* A_4, A_5 or S_4 (see e.g [3]).

The present work builds on the results of [4] to give a complete classification of which subgroups of the Möbius group Möb are realized as deck groups of iterates of a bicritical rational map. Our main results are the following two theorems.

Theorem A. *Let f be a bicritical map of odd degree d.*

1. *Then* $\operatorname{Deck}(f^k) \cong \mathbb{Z}_d$ *for all $k \in \mathbb{N}$ if and only if f is* not *a power map.*

(2) *Furthermore, f is a power map if and only if there exists $k \in \mathbb{N}$ such that* $\mathrm{Deck}(f^k) \cong \mathbb{Z}_n$ *for some $n > d$.*

Theorem B. *Let f be a bicritical map of even degree d.*

(1) *For each $k \in \mathbb{N}$,* $\mathrm{Deck}(f^k)$ *is isomorphic to either D_{2d}, D_{4d}, or $\mathbb{Z}_{d^n}$ for some $n \geq 1$.*

(2) *Furthermore, if f is not a power map then* $|\mathrm{Deck}(f^k)| \leq 4d$.

Additionally, in Proposition 6.1 we give examples showing that when d is even, each of the groups D_{2d} and D_{4d} are realized as $\mathrm{Deck}(f^k)$ for some degree d bicritical rational map f and some $1 \leq k \leq 3$, making the result of Theorem B sharp. A key step toward proving Theorems A and B is the following result.

Theorem 1.2. *Let f be a bicritical rational map and $\phi \in \mathrm{Deck}(f^k)$ for some k. Then $\phi(\mathcal{C}_f) = \mathcal{C}_f$.*

In [6], Pakovich studies the groups[1] $\mathrm{Deck}_\infty(f) := \bigcup_{k=1}^{\infty} \mathrm{Deck}(f^k)$ for rational maps f. He shows that, if f is not a power map, then $|\mathrm{Deck}_\infty(f)|$ is bounded, and this bound depends only on the degree d of the map f. A study of rational maps of minimal degree with a given deck group[2] was carried out in [2].

This paper is structured as follows. In Section 2 we give the required background on deck groups and Möbius transformations, as well as give sketch proofs of some important results from [4]. In Section 3, we give a proof of Theorem 1.2; the main difficulty is to prove the result in the quadratic case. After this, in Section 4 we will see that Theorem 1.2 implies a number of important results about deck groups of bicritical rational maps. In Section 5, given a bicritical rational map f, we study the Möbius transformations μ which satisfy $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$. The results of Sections 4 and 5 will then allow us to prove Theorems A and B in Section 6.

2. Background

For background on the dynamics of bicritical rational maps, we refer the reader to [5]. It can easily be verified that a bicritical rational map necessarily satisfies $|\mathcal{V}_f| = 2$. By putting the critical points at 0 and ∞, any bicritical rational map of degree d is conjugate to a map of the form $z \mapsto \frac{\alpha z^d + \beta}{\gamma z^d + \delta}$. For such a map, the deck group is clearly generated by the map $z \mapsto e^{2\pi i/d} z$. Thus if f is bicritical of degree d, we have $\mathrm{Deck}(f) \cong \mathbb{Z}_d$.

Given a Möbius transformation $\phi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, denote by $\mathrm{Fix}(\phi)$ its set of fixed points. A non-identity Möbius transformation of finite order satisfies $|\mathrm{Fix}(\phi)| = 2$. The following lemma collects together some standard properties of deck groups. We denote by $\deg_f(z)$ the *local degree* of f at the point $z \in \hat{\mathbb{C}}$.

[1] Pakovich uses the notation $\Sigma(f^k)$ for $\mathrm{Deck}(f^k)$ and $\Sigma_\infty(f)$ for $\mathrm{Deck}_\infty(f)$.

[2] In [2], elements of the deck group were called *half-symmetries*.

Lemma 2.1. *Let f be a rational map of degree $d \geq 1$.*

(1) *The group* $\mathrm{Deck}(f)$ *is finite, and so must be cyclic, dihedral, or isomorphic to one of the polyhedral groups A_4, A_5 or S_4. Furthermore* $|\mathrm{Deck}(f)| \leq d$.
(2) *Any non-identity element of* $\mathrm{Deck}(f)$ *has exactly two fixed points.*
(3) *If $z \in \hat{\mathbb{C}}$ and $\phi \in \mathrm{Deck}(f)$, then* $\deg_f(z) = \deg_f(\phi(z))$.
(4) *For all $k \geq 1$,* $\mathrm{Deck}(f^k) \subseteq \mathrm{Deck}(f^{k+1})$.
(5) *If f is bicritical of degree d, then* $\mathrm{Deck}(f) \cong \mathbb{Z}_d$. *Furthermore, each non-identity $\phi \in \mathrm{Deck}(f)$ has* $\mathrm{Fix}(\phi) = \mathcal{C}_f$.

We will make use of the following classical characterization of commuting Möbius transformations.

Lemma 2.2 ([1], Theorem 4.3.6)**.** *Let ϕ and μ be non-identity Möbius transformations. Then the following are equivalent.*

(1) $\phi \circ \mu = \mu \circ \phi$
(2) $\phi(\mathrm{Fix}(\mu)) = \mathrm{Fix}(\mu)$ *and* $\mu(\mathrm{Fix}(\phi)) = \mathrm{Fix}(\phi)$.
(3) *Either*
    (a) $\mathrm{Fix}(\mu) = \mathrm{Fix}(\phi)$, *or*
    (b) *ϕ, μ and $\phi \circ \mu$ are involutions and* $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\mu) = \varnothing$.

We now recall some results from [4] and sketch their proofs. The interested reader may refer to [4] for further details.

Proposition 2.3 ([4])**.** *Let f be a bicritical rational map of degree d, and let p be a prime number that does not divide d. Then for all natural numbers k, the group* $\mathrm{Deck}(f^k)$ *has no element of order p.*

Sketch Proof. If some element $\tau \in \mathrm{Deck}(f^k)$ were to have order p, then each element in $\hat{\mathbb{C}}$ would have orbit of length 1 or p under the action of $\langle \tau \rangle$. In particular the fiber $f^{-k}(w)$ over a regular value w contains d^k points, and since p does not divide d^k, the element τ must fix at least one element in such a fiber. But then as there are infinitely many regular points for f^k, we see that τ must be the identity, a contradiction. □

Theorem 2.4 ([4])**.** *Let f be a bicritical rational map and $k \in \mathbb{N}$. Then* $\mathrm{Deck}(f^k)$ *is either cyclic or dihedral. Furthermore, if the degree of f is odd, then* $\mathrm{Deck}(f^k)$ *is cyclic.*

Sketch Proof. The polyhedral groups A_4, A_5 and S_4 all contain elements of order 2 and elements of order 3. Thus if $\mathrm{Deck}(f^k)$ were polyhedral we would have $\deg(f) = d \geq 6$ by Proposition 2.3. But then by Lemma 2.1, then group $\mathrm{Deck}(f)$ would contain an element of order $d \geq 6$. But none of the polyhedral groups contain an element of order ≥ 6, so this is a contradiction.

Now suppose the degree of d is odd. In that case, 2 does not divide d and so by Proposition 2.3, $\mathrm{Deck}(f^k)$ cannot contain any element of order 2. Thus $\mathrm{Deck}(f^k)$ is not dihedral. □

We remark that the following result, which is a special case of Theorem B, was obtained in [4]. The results in this paper are obtained by methods markedly different from those employed in [4].

Theorem 2.5 ([4]). *If f is quadratic then the possibilities for* $\mathrm{Deck}(f^k)$ *(up to isomorphism) are $\mathbb{Z}_{2^n}$ for some $n \geq 1$, the Klein* Vierergruppe V_4 *or the dihedral group D_8 of order 8. Furthermore, if f is not a power map then* $|\mathrm{Deck}(f^k)| \leq 8$.

3. Proof of Theorem 1.2

The primary goal of this section is to prove Theorem 1.2, which we reproduce here for convenience.

Theorem 1.2. *Let f be a bicritical rational map and $\phi \in \mathrm{Deck}(f^k)$ for some k. Then $\phi(\mathcal{C}_f) = \mathcal{C}_f$.*

We begin by proving two useful lemmas. First, Lemma 3.1 generalizes an argument from [4] (a similar result is also given by Pakovich in [6]).

Lemma 3.1. *Let f be a degree d bicritical rational map with critical point set $\mathcal{C}_f$ and critical value set $\mathcal{V}_f$. Then if ϕ is a Möbius transformation such that $\phi(\mathcal{C}_f) = \mathcal{C}_f$, then there exists a unique Möbius transformation μ such that $\mu \circ f = f \circ \phi$. Furthermore $\mu(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. Once we prove existence, the uniqueness will follow from the surjectivity of f. We first prove the existence result for the sepcial case $g(z) = z^d$. In this case, ϕ is a Möbius transformation such that $\phi(\mathcal{C}_g) = \mathcal{C}_g$ if and only if $\phi(z) = az^{\pm 1}$ for some $a \in \mathbb{C} \setminus \{0\}$. But then $g \circ \phi = a^d z^{\pm d}$, and so taking $\mu(z) = a^d z$ completes the proof for $g(z) = z^d$.

Now suppose that f is an arbitrary bicritical rational map of degree d. Then there exist Möbius transformations α and β such that $f = \alpha \circ g \circ \beta$, where $g(z) = z^d$. In particular $\beta(\mathcal{C}_f) = \mathcal{C}_g$ and $\alpha(\mathcal{V}_g) = \mathcal{V}_f$. Thus if ϕ fixes $\mathcal{C}_f$ as a set then $\phi' = \beta \circ \phi \circ \beta^{-1}$ fixes $\mathcal{C}_g$ as a set, and by the above there exists μ' such that $\mu' \circ g = g \circ \phi'$. Hence taking $\mu = \alpha \circ \mu' \circ \alpha^{-1}$, a simple calculation yields

$$\begin{aligned}
\mu \circ f &= \mu \circ (\alpha \circ g \circ \beta) \\
&= (\alpha \circ \mu' \circ \alpha^{-1}) \circ \alpha \circ g \circ \beta \\
&= \alpha \circ (\mu' \circ g) \circ \beta \\
&= \alpha \circ (f \circ \phi') \circ \beta \\
&= \alpha \circ f \circ (\beta \circ \phi \circ \beta^{-1}) \circ \beta \\
&= \alpha \circ g \circ \beta \circ \phi \\
&= f \circ \phi
\end{aligned}$$

as desired. The fact that $\mu(\mathcal{V}_F) = \mathcal{V}_F$ is clear. □

In particular, when the map ϕ in Lemma 3.1 belongs to $\mathrm{Deck}(f^k)$, we get the following.

Lemma 3.2. *Let f be a bicritical rational map, $k \geq 2$, and $\phi \in \mathrm{Deck}(f^k)$. If $\phi(\mathcal{C}_f) = \mathcal{C}_f$, then there exists a unique map $\phi_{k-1} \in \mathrm{Deck}(f^{k-1})$ such that $f \circ \phi_k = \phi_{k-1} \circ f$. Furthermore, $\phi_{k-1}(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. By Lemma 3.1, it suffices to show that $\phi_{k-1} \in \mathrm{Deck}(f^{k-1})$. To see this, consider the following diagram.

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\phi_k"] & \hat{\mathbb{C}} \arrow[r, "f^{k-1}"] \arrow[d, "\phi_{k-1}"] & \hat{\mathbb{C}} \arrow[d, "\mathrm{id}"] \\
\hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}} \arrow[r, "f^{k-1}"] & \hat{\mathbb{C}}
\end{tikzcd}$$

The large outermost rectangle commutes since $\phi_k \in \mathrm{Deck}(f^k)$, and the square on the left comes directly from Lemma 3.1. Therefore, the square on the right commutes as well. As a consequence, $\phi_{k-1} \in \mathrm{Deck}(f^{k-1})$. $\square$

Next, we enumerate some simple properties of dihedral groups that we will use later.

Lemma 3.3. *Let $n \geq 2$. Consider the presentation of the dihedral group*

$$D_{2n} = \langle R, F \mid R^n = F^2 = (RF)^2 = \mathrm{id} \rangle. \tag{1}$$

(1) *For each integer $c \geq 3$, D_{2n} has at most one cyclic subgroup of order c. Furthermore, any generator of such a cyclic group is a power of R.*
(2) *Let n be even and suppose $\alpha \in D_{2n}$ has order 2. Then there exists a subgroup Γ of D_{2n} such that $\alpha \in \Gamma$ and $\Gamma \cong V_4$.*

Remark 3.4. *Since we allow $n = 2$, we consider the* Klein Vierergruppe *V_4 to be dihedral, i.e. $V_4 = D_4$.*

Proof. The proof of (1) is a standard result about dihedral groups, and is left to the reader, who may wish to appeal to the characterization of D_{2n} as the group of symmetries of the regular n-gon. For (2), note first that the result trivially holds for the group V_4. So assume $n > 2$. Using the presentation (1), the center $Z(D_{2n})$ is isomorphic to $\mathbb{Z}_2$, and is generated by $\mu = R^{n/2}$. Thus if α is any other element of order 2 in D_{2n}, then $\Gamma = \{\mathrm{id}, \mu, \alpha, \mu\alpha = \alpha\mu\}$ forms a subgroup of D_{2n} isomorphic to V_4. $\square$

3.1. **Proof of Theorem 1.2 for degree $d \geq 3$.** We now turn our attention to proving Theorem 1.2. The proof in the case for $\deg(f) \geq 3$ is relatively simple.

Lemma 3.5. *Let f be a bicritical rational map of degree $d \geq 3$ and suppose $\phi \in \mathrm{Deck}(f^k)$ for some k. Then $\phi(\mathcal{C}_f) = \mathcal{C}_f$.*

Proof. By Lemma 2.1, $\mathrm{Deck}(f) \cong \mathbb{Z}_d$ and is generated by the d-fold rotation ρ which fixes the points of $\mathcal{C}_f$ pointwise. By Theorem 2.4, $\mathrm{Deck}(f^k)$ is either cyclic or dihedral.

If $\mathrm{Deck}(f^k)$ is cyclic, then all non-identity elements of $\mathrm{Deck}(f^k)$ have the same pair of fixed points. Since by Lemma 2.1 we have $\mathrm{Deck}(f^k) \subseteq \mathrm{Deck}(f)$, we see that $\rho \in \mathrm{Deck}(f^k)$. Thus for all non-identity $\phi \in \mathrm{Deck}(f^k)$, we have $\mathrm{Fix}(\phi) = \mathrm{Fix}(\rho) = \mathcal{C}_f$.

If $\mathrm{Deck}(f^k)$ is dihedral, then Lemma 3.3 part (1) implies that $\mathrm{Deck}(f)$ is the unique cyclic subgroup of order d in $\mathrm{Deck}(f^k)$, and the generator ρ of $\mathrm{Deck}(f)$ is an iterate of an element of maximal order in $\mathrm{Deck}(f^k)$. That is, using the presentation (1), $\rho = R^\ell$ for some $\ell \geq 1$. Hence all powers of R fix the set $\mathrm{Fix}(\rho) = \mathcal{C}_f$. Since a generator F as in (1) is an involution, it must either swap the two critical points or fix them pointwise; either way F fixes $\mathcal{C}_f$. Since the generators of $\mathrm{Deck}(f^k)$ fix $\mathcal{C}_f$ as a set, it follows that all elements of $\mathrm{Deck}(f^k)$ must fix $\mathcal{C}_f$ as a set. □

3.2. **Proof of Theorem 1.2 in the quadratic case.** In the quadratic case a more careful analysis is required, since if $\mathrm{Deck}(f^k)$ is dihedral, it does not immediately follow that any order 2 subgroup of $\mathrm{Deck}(f^k)$ must be the group $\mathrm{Deck}(f)$. To study this case, we introduce terminology from [4].

Definition 3.6 ([4]). *We say a bicritical rational map with critical values v_1 and v_2 is* critically coalescing *if $f(v_1) = f(v_2)$.*

We first give some simple properties of critically coalescing maps.

Lemma 3.7. *Let f be a degree d critically coalescing rational map with critical points $\mathcal{C}_f = \{c_1, c_2\}$ and critical values $\mathcal{V}_f = \{v_1, v_2\}$, with $f(c_1) = v_i$ for $i = 1, 2$. Then*

1. *$\mathcal{C}_f \cap \mathcal{V}_f = \varnothing$.*
2. *$f^k(c_1) = f^k(c_2)$ for all $k \geq 2$*

Proof.

1. Without loss of generality suppose $v_1 \in \mathcal{C}_f \cap \mathcal{V}_f$. Then $\{v_1, f(v_2)\} \subseteq f^{-1}(f(v_1))$, so $f(v_1)$ has $d+1$ preimages, counting multiplicity. This is a contradiction.
2. This follows easily from the part (i) and the fact that $f(v_1) = f(v_2)$. □

The next lemma is a slight generalization of a result from [4]. It relates Definition 3.6 with the groups $\mathrm{Deck}(f^k)$ for a bicritical rational map.

Lemma 3.8. *Let f be a bicritical rational map of even degree d. If* $\mathrm{Deck}(f^k)$ *is dihedral for some $k \in \mathbb{N}$, then f is critically coalescing.*

Proof. Write $\mathrm{Deck}(f) = \langle \tau \rangle$ and suppose $\mu = \tau^{d/2}$ is the unique element of order 2 in $\mathrm{Deck}(f)$. Suppose $k > 1$ is minimal such that $\mathrm{Deck}(f^k)$ is dihedral. By Lemma 3.3, there exists Γ, a subgroup of $\mathrm{Deck}(f^k)$ such that $\Gamma \cong V_4$

and $\mu \in \Gamma$. Write $\Gamma = \{\mathrm{id}, \mu, \alpha, \beta\}$, where α and β are order 2 elements of $\mathrm{Deck}(f^k)$. Since Γ is abelian and $\mathrm{Fix}(\mu) = \mathcal{C}_f$, then by Lemma 2.2 we have $\alpha(\mathcal{C}_f) = \beta(\mathcal{C}_f) = \mathcal{C}_f$. Thus, by Lemma 3.2, there exists $\nu \in \mathrm{Deck}(f^{k-1})$ such that $\nu \circ f = f \circ \alpha$. Furthermore, $\nu(\mathcal{V}_f) = \mathcal{V}_f = \{v_1, v_2\}$. Since α is an order 2 element of $\mathrm{Deck}(f^k)$ distinct from μ, it cannot be an element of $\mathrm{Deck}(f)$. Hence $f \circ \alpha \neq f$, and so $\nu \neq \mathrm{id}$. By the assumption on the minimality of k, $\mathrm{Deck}(f^{k-1})$ must be cyclic. Since all non-identity elements of a finite cyclic group of Möbius transformations share the same pair of fixed points, we see that for all non-identity elements $\gamma \in \mathrm{Deck}(f^{k-1})$ we have $\mathrm{Fix}(\gamma) = \mathrm{Fix}(\mu) = \mathcal{C}_f$. In particular $\mathrm{Fix}(\nu) = \mathcal{C}_f$.

Suppose that ν fixes the elements of $\mathcal{V}_f$ pointwise. Then we have $\mathcal{C}_f = \mathcal{V}_f$, and so f is a power map. But this is impossible, since $\mathrm{Deck}(f^k)$ is always cyclic for power maps. So ν must swap the elements of $\mathcal{V}_f$, and so ν is an involution. But since $\mathrm{Deck}(f^{k-1})$ is cyclic, it contains at most one involution. But since μ is an order 2 element of $\mathrm{Deck}(f) \subseteq \mathrm{Deck}(f^{k-1})$, me must have $\nu = \mu$. Thus $\nu = \mu \in \mathrm{Deck}(f)$ interchanges the elements of $\mathcal{V}_f$, and so $f(v_1) = f(v_2)$. □

It was shown in [4] that the converse to the above lemma is true in the quadratic case. However, in Example 4.7 we will show that for higher degrees, the converse to Lemma 3.8 does not hold.

The following is a simple topological observation.

Lemma 3.9. *Let F be a rational map of degree d and suppose* $\mathrm{Deck}(F)$ *contains an element of order k. Then there exists $z \in \hat{\mathbb{C}}$ such that* $\deg_F(z) \geq k$.

Proof. First notice that if ϕ has order k, then for every $\zeta \notin \mathrm{Fix}(\phi)$, the points $\zeta, \phi(\zeta), \dots, \phi^{k-1}(\zeta)$ must all be distinct. Otherwise, if $\phi^i(\zeta) = \phi^j(\zeta)$ for some $0 \leq i < j \leq k-1$, then ϕ^{j-i} fixes the point $\phi^i(\zeta)$. But then $\mathrm{Fix}(\phi^{j-i}) \supseteq \mathrm{Fix}(\phi) \cup \{\phi^i\}$, so that $|\mathrm{Fix}(\phi^{j-1})| \geq 3$. This implies $\phi^{j-i} = \mathrm{id}$. Since $j - i < k$, this is a contradiction.

Let $z \in \mathrm{Fix}(\phi)$. Suppose that V is a simply connected neighborhood of $F(z)$ such that $V \cap \mathcal{V}_F \subseteq \{F(z)\}$, and let U be the component of $F^{-1}(V)$ which contains z. By restricting V if necessary, we may assume that $F^{-1}(F(z)) \cap U = \{z\}$ and that z is the only element of $\mathrm{Fix}(\phi)$ in U.

Let $w \neq F(z)$ be an element of V. Then there exists $u_0 \in U$ such that $F(u_0) = w$. For each $0 \leq j \leq k-1$, define $u_j = \phi^j(u_0)$. From the above, we know that $u_i \neq u_j$ for $i \neq j$. Furthermore, since $F = F \circ \phi^j$ and $\phi^j(z) = z$ for all $0 \leq j \leq k-1$, we see that $u_j \in U$ for all $0 \leq j \leq k-1$. Hence $F\colon U \to V$ is a branched covering of degree (at least) k on the simply connected set U. Since the only critical point in U is z, we have $\deg_F(z) \geq k$ by the Riemann-Hurwitz Theorem. □

We are particularly interested in applying the previous result to the case where $F = f^k$ is an iterate of a quadratic rational map f and $\mathrm{Deck}(f^k)$ is dihedral.

Lemma 3.10. *Let f be a quadratic rational map and suppose* $\mathrm{Deck}(f^k)$ *is dihedral. If f^k has a critical point with local degree greater than 2, then one of the critical points c_1 of f is periodic of some period p. Furthermore:*

- *the second critical point c_2 satisfies $f^p(c_2) = c_1$ and*
- *for either critical point c, $f^n(c) = c_1$ if and only if $n = ap$ for some $a \geq 0$.*

Proof. By Lemma 3.8, f must be critically coalescing. Let $\mathcal{C}_f = \{c_1, c_2\}$. Since f is critically coalescing, we see that by Lemma 3.7 that $f(c_i) \notin \mathcal{C}_f$ for $i = 1, 2$.

A point $z \in \hat{\mathbb{C}}$ maps forward with local degree greater than 1 under f^k if and only if z is a preimage $f^{-j}(c_i)$ for some $0 \leq j < k$ and $i = 1, 2$. Furthermore, if z maps forward by local degree strictly greater than 2 under f^k, then the forward orbit

$$\mathcal{O}_k(z) = (z, f(z), f^2(z), \ldots, f^{k-1}(z))$$

must contain (at least) two critical points of f. If the same critical point c_i appears twice, we are done, since then that critical point would be periodic. So assume without loss of generality that there exist $0 \leq n < m < k$ with $f^n(z) = c_2$ and $f^m(z) = c_1$. Then we have $f^{m-n}(c_2) = c_1$. But since f is critically coalescing, then by Lemma 3.7 we have $f^\ell(c_1) = f^\ell(c_2)$ for all $\ell \geq 2$. Thus $f^{m-n}(c_1) = c_1$ and so c_1 is a periodic critical point under f.

Now suppose that p is the period of c_1, so that $p > 0$ is minimal such that $f^p(c_1) = c_1$. Since f is critically coalescing, we also have $f^p(c_2) = c_1$, and if there were $0 < j < p$ such that $f^j(c_2) = c_1$, then this would imply $f^j(c_1) = c_1$, which is a contradiction. □

Proposition 3.11. *Let f be a quadratic rational map and suppose that for some $k \in \mathbb{N}$ the group* $\mathrm{Deck}(f^k)$ *is dihedral. If $\phi \in$* $\mathrm{Deck}(f^k)$ *then $\phi(\mathcal{C}_f) = \mathcal{C}_f$.*

Proof. If $\mathrm{Deck}(f^k)$ is isomorphic to V_4, then $\mathrm{Deck}(f^k)$ is abelian. Thus every element of $\mathrm{Deck}(f^k)$ commutes with μ, the unique order 2 element of $\mathrm{Deck}(f)$. Hence by Lemma 2.2, since $\mathrm{Fix}(\mu) = \mathcal{C}_f$, we have $\phi(\mathcal{C}_f) = \mathcal{C}_f$ for all $\phi \in \mathrm{Deck}(f^k)$.

We now assume that $\mathrm{Deck}(f^k)$ is dihedral and contains an element of order $n > 2$. By Lemma 3.9, there must exist $z \in \hat{\mathbb{C}}$ such that $\deg_{f^k}(z) \geq n > 2$ and so by Lemma 3.10, f has a periodic critical point, c_1, and the other critical point c_2 eventually maps onto c_1, but is not in the forward orbit of c_1. We will show that the orbit $\mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1)$ under the action of $\mathrm{Deck}(f^k)$ is equal to $\mathcal{C}_f = \{c_1, c_2\}$. To see that $c_2 \in \mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1)$, let Γ be a subgroup of $\mathrm{Deck}(f^k)$ which is isomorphic to V_4 and which contains μ (such a subgroup exists by Lemma 3.3). By Lemma 2.2, every non-identity element $\phi \in \Gamma$ must have $\phi(\mathcal{C}_f) = \mathcal{C}_f$. In particular, if $\phi \neq \mu$, then since μ and ϕ are distinct and both have order 2, we see that ϕ must transpose the elements of $\mathrm{Fix}(\mu) = \mathcal{C}_f$. Hence $\phi(c_1) = c_2$, meaning $\{c_1, c_2\} = \mathcal{C}_f \subseteq \mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1)$.

To prove the reverse inclusion, suppose that $a \in \mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1)$, so that there exists $\phi \in \mathrm{Deck}(f^k)$ such that $\phi(c_1) = a$. Let $p \geq 2$ be the period of c_1. By Lemma 2.1, we have $\deg_{f^s}(a) = \deg(f^s)(c_1)$ for all $s \geq k$. Thus, for all $j \geq 0$ and $1 \leq m \leq p$ such that $jp + m \geq k$ we have

$$\deg_{f^{jp+m}}(a) = \deg_{f^{jp+m}}(c_1) = 2^{j+1}. \tag{2}$$

Since $\deg_{f^{jp+m}}(a) > 1$, it follows that a must eventually map onto a critical point. Let $q \geq 0$ be minimal such that $f^q(a) \in \mathcal{C}_f$.

We now show that $a \in \mathcal{C}_f$. Let j be minimal such that $jp + 1 \geq k$. Then by (2), we have $\deg_{f^{jp+1}}(a) = 2^{j+1}$ and so the orbit

$$\mathcal{O}_{jp}(a) = (a, f(a), f^2(a), \ldots, f^{jp}(a))$$

must contain exactly $j + 1$ critical points. Suppose $a \notin \mathcal{C}_f$ so that $q > 0$. By Lemma 3.10, the points $f^q(a), f^{p+q}(a), f^{2p+q}(a), \ldots$ are all critical. But since $q > 0$, the inequality $ip + q \leq jp$ has at most j solutions for $i \geq 0$. Thus there are at most j critical points in the orbit $\mathcal{O}_{jp}(a)$, which is a contradiction. Thus $q = 0$ and so $a \in \mathcal{C}_f$. Hence $\mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1) \subseteq \mathcal{C}_f$. We conclude that $\mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1) = \mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1) = \mathcal{C}_f$. It follows that $\phi(\mathcal{C}_f) = \mathcal{C}_f$ for all $\phi \in \mathrm{Deck}(f^k)$. □

Proof of Theorem 1.2. In light of Lemma 3.5, we only need to prove the result in the quadratic case. By Theorem 2.4, $\mathrm{Deck}(f^k)$ is either cyclic or dihedral. If $\mathrm{Deck}(f^k)$ is cyclic, then every non-identity element of $\mathrm{Deck}(f^k)$ has the same set of fixed points. But since the unique order 2 element $\mu \in \mathrm{Deck}(f)$ has $\mathrm{Fix}(\mu) = \mathcal{C}_f$. Since $\mu \in \mathrm{Deck}(f^k)$, we see that for all non-identity $\phi \in \mathrm{Deck}(f^k)$ we have $\mathrm{Fix}(\phi) = \mathcal{C}_f$. Hence in this case $\phi(\mathcal{C}_f) = \mathcal{C}_f$. On the other hand, if $\mathrm{Deck}(f^k)$ is dihedral then the result holds by Proposition 3.11. □

4. Consequences of Theorem 1.2

Theorem 1.2 has a number of useful consequences. We first state a strengthened version of Lemma 3.2.

Proposition 4.1. *Let f be a bicritical rational map and $\phi_k \in \mathrm{Deck}(f^k)$ for some k. Then there exists a unique $\phi_{k-1} \in \mathrm{Deck}(f^{k-1})$ such that $f \circ \phi_k = \phi_{k-1} \circ f$. Moreover $\phi_{k-1}(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. The proof is the same as Lemma 3.2, with the hypothesis that $\phi_k(\mathcal{C}_f) = \mathcal{C}_f$ removed by Theorem 1.2. □

We now use Proposition 4.1 to prove a number of preliminary results which we will use to prove the main theorems. Observe that by Proposition 4.1, if for some $k > 1$ we have $\phi_k \in \mathrm{Deck}(f^k)$, then we can recursively define a sequence

$$(\phi_k, \phi_{k-1}, \ldots, \phi_1, \phi_0 = \mathrm{id})$$

where for each j, $\phi_j \in \mathrm{Deck}(f^j)$ and $f \circ \phi_j = \phi_{j-1} \circ f$. Each ϕ_j is uniquely determined by the initial choice of ϕ_k, and we must have $f^{k-j} \circ \phi_k = \phi_j \circ f^{k-j}$. This gives the following commutative diagram.

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\phi_k"] & \hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\phi_{k-1}"] & \dots \arrow[r, "f"] & \hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\phi_1"] & \hat{\mathbb{C}} \arrow[d, "\mathrm{id}"] \\
\hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}} \arrow[r, "f"] & \dots \arrow[r, "f"] & \hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}}
\end{tikzcd}$$

In particular, if $j = 1$ we obtain the following result.

Proposition 4.2. *Let f be a bicritical rational map. Let $k > 1$ and suppose $\phi_k \in \mathrm{Deck}(f^k)$. Then there exists a unique $\phi_1 \in \mathrm{Deck}(f)$ such that the following diagram commutes.*

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "f^{k-1}"] \arrow[d, "\phi_k"'] & \hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\phi_1"] & \hat{\mathbb{C}} \arrow[d, "\mathrm{id}"] \\
\hat{\mathbb{C}} \arrow[r, "f^{k-1}"] & \hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}}
\end{tikzcd}$$

Furthermore

(1) *ϕ_1 is the identity if and only if $\phi_k \in \mathrm{Deck}(f^{k-1})$.*
(2) *$\phi_1(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. By Proposition 4.1 and the discussion following Proposition 4.1, we know that there exists a unique $\phi_1 \in \mathrm{Deck}(f)$ such that $f^{k-1} \circ \phi_k = \phi_1 \circ f^{k-1}$. This proves the diagram commutes. Now suppose that $\phi_k \in \mathrm{Deck}(f^{k-1})$. Then since $f^{k-1} = f^{k-1} \circ \phi_k$, we see from the diagram that $\phi_1 = \mathrm{id}$. On the other hand, if $\phi_1 = \mathrm{id}$, then the diagram shows that $f^{k-1} \circ \phi_k = f^{k-1}$, and so $\phi_k \in \mathrm{Deck}(f^{k-1})$. The assertion that $\phi_1(\mathcal{V}_f) = \mathcal{V}_f$ again follows from Proposition 4.1. $\square$

It should be noted that in general, an element $\phi \in \mathrm{Deck}(f)$ need not map $\mathcal{V}_f$ to itself. For example, if $f(z) = \frac{1}{z^2-1}$, then the unique non-identity element of $\mathrm{Deck}(f)$ is $\phi(z) = -z$, which fixes the critical points 0 and ∞ of f. However, $\mathcal{V}_f = \{0, -1\}$, which is clearly not preserved by ϕ.

The following can be thought of as a partial converse to Proposition 4.1.

Lemma 4.3. *Let f be a bicritical rational map of degree d and suppose μ is a Möbius transformation such that $\mu(\mathcal{V}_f) = \mathcal{V}_f$. Then there exists a Möbius transformation ϕ such that $f \circ \phi = \mu \circ f$ and $\phi(\mathcal{C}_f) = \mathcal{C}_f$. In particular, if $\mu \in \mathrm{Deck}(f^k)$ for some $k \geq 1$ then $\phi \in \mathrm{Deck}(f^{k+1})$.*

Proof. The proof proceeds like that of Lemma 3.1, but note in this case there is no uniqueness. First, suppose $g(z) = z^d$. Then if $\mu(\mathcal{V}_g) = \mathcal{V}_g = \{0, \infty\}$, we have $\mu(z) = az^{\pm 1}$ for some $a \in \mathbb{C} - \{0\}$. Thus taking $\phi(z) = a^d z^{\pm 1}$, we get $g \circ \phi = \mu \circ g$ and $\phi(\mathcal{C}_g) = \mathcal{C}_g$ as required.

For the general case, we again note that if f is a bicritical rational map of degree d, then there exist Möbius transformations α and β such that $f = \alpha \circ g \circ \beta$ for $g(z) = z^d$, so that $\alpha(\mathcal{V}_g) = \mathcal{V}_f$ and $\beta(\mathcal{C}_f) = \mathcal{C}_g$. Thus if

μ is a Möbius transformation and $\mu(\mathcal{V}_f) = \mathcal{V}_f$, then $\mu' = \alpha^{-1} \circ \mu \circ \alpha$ must satisfy $\mu'(\mathcal{V}_g) = \mathcal{V}_g$. Hence, by the previous paragraph, there exists ϕ' such that $g \circ \phi' = \mu' \circ g$ and $\phi'(\mathcal{C}_g) = \mathcal{C}_g$. Thus taking $\phi = \beta^{-1} \circ \phi' \circ \beta$ we get $f \circ \phi = \mu \circ f$ and $\phi(\mathcal{C}_f) = \mathcal{C}_f$, as required.

Finally, if $\mu \in \mathrm{Deck}(f^k)$ then since $f \circ \phi = \mu \circ f$, composing on the left by f^k gives

$$f^{k+1} \circ \phi = f^k \circ (f \circ \phi) = f^k \circ (\mu \circ f) = (f^k \circ \mu) \circ f = f^k \circ f = f^{k+1}$$

and so $\phi \in \mathrm{Deck}(f^{k+1})$. □

Lemma 4.4. *Let f be a bicritical rational map. Suppose for some k that* $\mathrm{Deck}(f^k) = \mathrm{Deck}(f^{k+1})$. *Then* $\mathrm{Deck}(f^{k+2}) = \mathrm{Deck}(f^{k+1}) = \mathrm{Deck}(f^k)$.

Proof. Let $\phi \in \mathrm{Deck}(f^{k+2})$. By Proposition 4.1, there exists $\mu \in \mathrm{Deck}(f^{k+1})$ such that

$$f \circ \phi = \mu \circ f. \tag{3}$$

Since $\mathrm{Deck}(f^{k+1}) = \mathrm{Deck}(f^k)$, we see that $\mu \in \mathrm{Deck}(f^k)$. But then post-composing (3) by f^k yields

$$f^{k+1} \circ \phi = f^k \circ (\mu \circ f) = (f^k \circ \mu) \circ f = f^{k+1}$$

and so $\phi \in \mathrm{Deck}(f^{k+1})$. □

Remark 4.5. *The authors are unaware if there are counterexamples to Lemma 4.4 in the case where f is a general rational map. That is, if f is a rational map and $k \geq 1$, is it true that* $\mathrm{Deck}(f^k) = \mathrm{Deck}(f^{k+1})$ *implies* $\mathrm{Deck}(f^n) = \mathrm{Deck}(f^k)$ *for all $n \geq k$?*

In the following, we will use the notation $\mathrm{Deck}^*(f^k) = \mathrm{Deck}(f^k) \backslash \mathrm{Deck}(f^{k-1})$, with the convention that $\mathrm{Deck}(f^0) = \mathrm{Deck}^*(f^0) = \{\mathrm{id}\}$. Note that by using this convention, we have $\mathrm{Deck}^*(f) = \mathrm{Deck}(f) \setminus \{\mathrm{id}\}$ is the set of non-identity elements of $\mathrm{Deck}(f)$.

Lemma 4.6. *Let f be a bicritical rational map and $k \geq 0$. Then* $\mathrm{Deck}^*(f^{k+1}) \neq \varnothing$ *if and only if there exists* $\mu \in \mathrm{Deck}^*(f^k)$ *such that* $\mu(\mathcal{V}_f) = \mathcal{V}_f$.

Proof. Suppose $\phi \in \mathrm{Deck}^*(f^{k+1})$. By Proposition 4.1, there exists $\mu \in \mathrm{Deck}(f^k)$ such that

$$\mu \circ f = f \circ \phi \tag{4}$$

and $\mu(\mathcal{V}_f) = \mathcal{V}_f$. If μ is not an element of $\mathrm{Deck}^*(f^k)$, then $\mu \in \mathrm{Deck}(f^{k-1})$. Thus $f^{k-1} \circ \mu = f^{k-1}$, and so composing f^{k-1} on the left of (4) we get

$$f^k = f^{k-1} \circ \mu \circ f = f^k \circ \phi$$

and so $\phi \in \mathrm{Deck}(f^k)$. But this contradicts $\phi \in \mathrm{Deck}^*(f^{k+1})$, and so we conclude that $\mu \in \mathrm{Deck}^*(f^k)$.

Conversely, suppose that there exists $\mu \in \mathrm{Deck}^*(f^k)$ such that $\mu(\mathcal{V}_f) = \mathcal{V}_f$. It follows from Lemma 4.3 that there exists $\phi \in \mathrm{Deck}(f^{k+1})$ such that $\mu \circ f = f \circ \phi$. Suppose that $\phi \in \mathrm{Deck}(f^k)$. Then Proposition 4.1 asserts that there is a unique $\mu' \in \mathrm{Deck}(f^{k-1})$ such that $\mu' \circ f = f \circ \phi$. But then

$\mu = \mu'$, and so this contradicts $\mu \in \mathrm{Deck}^*(f^k)$. Hence $\phi \in \mathrm{Deck}^*(f^{k+1})$ and so $\mathrm{Deck}^*(f^{k+1}) \neq \varnothing$. $\square$

Before continuing, we provide an example which shows that the converse to Lemma 3.8 does not generalize to higher degrees.

Example 4.7. Let $f(z) = \frac{z^4-1}{z^4+i}$. Then $\mathcal{V}_f = \{1, i\}$ and $f(1) = f(i) = 0$; thus f is critically coalescing. One can check by direct computation that $\mathrm{Deck}(f^2) = \mathrm{Deck}(f) \cong \mathbb{Z}_4$ and then appeal to Lemma 4.4, but here is an argument that also makes use of Lemma 4.6. It is easy to see that $\mathrm{Deck}(f)$ is generated by the order 4 rotation $\rho(z) = iz$. But then for all $n \in \{1, 2, 3\}$, we have $\rho^n(\mathcal{V}_f) \neq \mathcal{V}_f$, and so there does not exist a non-identity element of $\mathrm{Deck}(f)$ which fixes $\mathcal{V}_f$ as a set. From Lemma 4.6, we see that $\mathrm{Deck}(f^2) = \mathrm{Deck}(f) \cong \mathbb{Z}_4$. Then by Lemma 4.4, we see that $\mathrm{Deck}(f^k) \cong \mathbb{Z}_4$ for all $k \geq 1$.

Lemma 4.8. *Let f be a bicritical rational map of degree $d \geq 2$. Then*

$$\frac{|\mathrm{Deck}(f^k)|}{|\mathrm{Deck}(f^{k-1})|} \leq d. \tag{5}$$

Furthermore, if f is not a power map, then the quotient is at most 2.

Proof. Suppose $\phi \in \mathrm{Deck}(f^k)$. Then by Proposition 4.2, there exists a unique $\mu \in \mathrm{Deck}(f)$ such that $f^{k-1} \circ \phi = \mu \circ f^{k-1}$ and $\mu \neq \mathrm{id}$ if and only if $\phi \in \mathrm{Deck}^*(f^k)$. Furthermore, $\mu(\mathcal{V}_f) = \mathcal{V}_f$.

Define $h\colon \mathrm{Deck}(f^k) \to \mathrm{Deck}(f)$ by $h(\phi) = \mu$, where μ is defined as the map from the above paragraph. We claim that h is a homomorphism. To see this, note that if $f^{k-1} \circ \phi_1 = \mu_1 \circ f^{k-1}$ and $f^{k-1} \circ \phi_2 = \mu_2 \circ f^{k-1}$, then

$$\begin{aligned} f^{k-1} \circ \phi_1 \circ \phi_2 &= \mu_1 \circ f^{k-1} \circ \phi_2 \\ &= \mu_1 \circ \mu_2 \circ f^{k-1} \\ &= \mu_1 \circ \mu_2 \circ f^{k-1} \end{aligned}$$

It follows that $h(\phi_1 \circ \phi_2) = h(\phi_1) \circ h(\phi_2)$. By Proposition 4.2, we have $\ker h = \mathrm{Deck}(f^{k-1})$. Since each coset of $\ker h$ in $\mathrm{Deck}(f^k)$ has cardinality equal to $|\mathrm{Deck}(f^{k-1})|$, and since there are at most $|\mathrm{Deck}(f)| = d$ cosets, we conclude from the First Isomorphism Theorem that (5) holds.

To prove the final claim, recall that if f is bicritical then all non-identity elements $\mu \in \mathrm{Deck}(f)$ satisfy $\mathrm{Fix}(\mu) = \mathcal{C}_f$. Furthermore, if $\mu(\mathcal{V}_f) = \mathcal{V}_f$, then μ either fixes the elements of $\mathcal{V}_f$ pointwise, or μ is an involution which transposes the elements of $\mathcal{V}_f$. Since an involution is completely defined by its fixed points, we see that there is at most one $\mu \in \mathrm{Deck}(f)$ which transposes the elements of $\mathcal{V}_f$. Denote this element by ν.

Now suppose f is bicritical but not a power map, so that $|\mathcal{C}_f \cup \mathcal{V}_f| \geq 3$. If $\mu \in \mathrm{Deck}(f)$ fixes the elements of $\mathcal{V}_f$ pointwise, then $|\mathrm{Fix}(\phi)| \geq 3$, so that ϕ is the identity. It follows that $\operatorname{ran} h \subseteq \{\mathrm{id}, \nu\}$, and so again by the First Isomorphism Theorem we must have that the quotient (5) is at most 2. $\square$

To end this section, we give a result of independent interest. A general form of the following result was proved by Pakovich in [6], making use of algebraic curves. Here we give a dynamical proof, using properties of deck groups.

Proposition 4.9. *Let f be a bicritical rational map and $\phi \in \operatorname{Deck}(f^k)$. Then $f \circ \phi$ is conjugate to f.*

Proof. Let $\phi_k = \phi$. By Proposition 4.1, there exists $\phi_{k-1} \in \operatorname{Deck}(f^{k-1})$ such that $f \circ \phi_k = \phi_{k-1} \circ f$. Precomposing by ϕ_{k-1} gives

$$f \circ \phi_k \circ \phi_{k-1} = \phi_{k-1} \circ f \circ \phi_{k-1}. \tag{6}$$

Again by Proposition 4.1, there exists $\phi_{k-2} \in \operatorname{Deck}(f^{k-2})$ such that $f \circ \phi_{k-1} = \phi_{k-2} \circ f$, and so (6) becomes

$$f \circ \phi_k \circ \phi_{k-1} = \phi_{k-1} \circ f \circ \phi_{k-1} = \phi_{k-1} \circ \phi_{k-2} \circ f.$$

We can repeat the above process to recursively obtain $\phi_j \in \operatorname{Deck}(f^j)$, so that

$$f \circ \phi_k \circ \phi_{k-1} \cdots \circ \phi_1 = \phi_{k-1} \circ \cdots \phi_1 \circ \phi_0 \circ f. \tag{7}$$

Furthermore, we must have $\phi_0 = \mathrm{id}$, so denoting $\Phi = \phi_{k-1} \circ \cdots \phi_1$, we see that (7) becomes

$$(f \circ \phi) \circ \Phi = \Phi \circ f.$$

Since Φ is a Möbius transformation, the result follows. □

5. Möbius transformations preserving the sets of critical points and critical values values of a bicritical rational map

As can be ascertained from Proposition 4.1 and Lemma 4.6, the Möbius transformations μ such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$ are of particular importance when it comes to analyzing the groups $\operatorname{Deck}(f^k)$. In fact, when f is not a power map these two conditions on μ are very restrictive.

Lemma 5.1. *Let f be a bicritical rational map of degree d such that $|\mathcal{C}_f \cup \mathcal{V}_f| = 3$. Then the only Möbius transformation μ satisfying $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$ is the identity. Furthermore,* $\operatorname{Deck}(f^k) \cong \mathbb{Z}_d$ *for all $k \geq 1$*

Proof. Since $|\mathcal{C}_f \cup \mathcal{V}_f| = 3$, there exists a unique $w \in \mathcal{C}_f \cap \mathcal{V}_f$. But then any Möbius transformation μ such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$ must fix w. Therefore, μ would have to act as the identity on the three element set $\mathcal{C}_f \cup \mathcal{V}_f$, and so $\mu = \mathrm{id}$. Since $\operatorname{Deck}(f) \cong \mathbb{Z}_d$, the final claim then follows from Lemmas 4.4 and 4.6. □

We now consider the case where f is bicritical and $\mathcal{C}_f \cap \mathcal{V}_f = \varnothing$. This is equivalent to $|\mathcal{C}_f \cup \mathcal{V}_f| = 4$. First we consider the set of Möbius transformations μ such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$ in this case.

Lemma 5.2. *Let f be a bicritical rational map such that $|\mathcal{C}_f \cup \mathcal{V}_f| = 4$. Then there exist at most four Möbius transformations μ such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$, which are the following.*

(1) *μ fixes the elements of $\mathcal{C}_f$ and $\mathcal{V}_f$ pointwise, so that $\mu = \mathrm{id}$.*
(2) *μ_1 such that $\mathrm{Fix}(\mu_1) = \mathcal{C}_f$ and μ_1 swaps the elements of $\mathcal{V}_f$.*
(3) *μ_2 such that $\mathrm{Fix}(\mu_2) = \mathcal{V}_f$ and μ_2 swaps the elements of $\mathcal{C}_f$.*
(4) *μ_3 such that μ_3 swaps the elements of $\mathcal{C}_f$ and swaps the elements of $\mathcal{V}_f$.*

Furthermore, each μ_i, $i = 1, 2, 3$ is an an involution. Furthermore, if all the above maps exist for the map f, they form a group which is isomorphic to V_4.

Proof. Since a Möbius transformation is uniquely characterized by its action on three points, we see there are the following possibilities for μ.

Furthermore, since each of the maps μ_1, μ_2 and μ_3 have a period 2 orbit, each one must be an involution. To complete the proof, we need to show that the four Möbius transformations listed above form a group. But by Lemma 2.2, μ_1 and μ_2 commute. Furthermore, $\mu_1 \circ \mu_2$ is an involution which swaps the elements of $\mathcal{C}_f$ and swaps the elements of $\mathcal{V}_f$, so that $\mu_1 \circ \mu_2 = \mu_3$. Thus $\langle \mu_1, \mu_2 \rangle = \{\mathrm{id}, \mu_1, \mu_2, \mu_3\} \cong V_4$. □

We will continue to use the notation μ_i, $i = 1, 2, 3$ to denote the transformations obtained from the above lemma. Since any element of $\mathrm{Deck}(f)$ fixes $\mathcal{C}_f$ pointwise, we have $\mu_2, \mu_3 \notin \mathrm{Deck}(f)$.

Lemma 5.3. *Let f be a bicritical rational map of degree d, and suppose f is not a power map. If $\mathrm{Deck}^*(f^2) \neq \varnothing$ then $\mathrm{Deck}(f^2) \cong D_{2d}$.*

Proof. By Lemma 4.8, since f is not a power map we have $|\mathrm{Deck}(f^2)| = 2|\mathrm{Deck}(f)| = 2d$. Since $\mathrm{Deck}(f) \cong \mathbb{Z}_d$, it follows from Theorem 2.4 that $\mathrm{Deck}(f^2) \cong \mathbb{Z}_{2d}$ or $\mathrm{Deck}(f^2) \cong D_{2d}$. Furthermore, by assumption $\mathrm{Deck}(f^2) \neq \mathrm{Deck}(f)$, and so we know from Lemma 4.6 that there exists a non-identity $\mu \in \mathrm{Deck}(f)$ such that $\mu(\mathcal{V}_f) = \mathcal{V}_f$. By Lemma 5.1 and the fact that f is not a power map, we must have $|\mathcal{C}_f \cup \mathcal{V}_f| = 4$, and so μ must be the map μ_1 from Lemma 5.2. Since μ_1 swaps the elements of $\mathcal{V}_f = \{v_1, v_2\}$ we see that $f(v_1) = f(v_2)$, and so f is critically coalescing.

Now suppose that $\mathrm{Deck}(f^2) \cong \mathbb{Z}_{2d}$. By Lemma 3.9, there exists $c \in \hat{\mathbb{C}}$ such that $\deg_{f^2}(c) \geq 2d$. But as f is bicritical, we must have $\deg_{f^2}(c) \in \{1, d, d^2\}$. Hence $\deg_{f^2}(c) = d^2$ and so $c \in \mathcal{C}_f \cap \mathcal{V}_f$. Therefore by Lemma 5.1, we have $\mathrm{Deck}(f^2) \cong \mathbb{Z}_d$, which is a contradiction. Thus $\mathrm{Deck}(f^2) \cong D_{2d}$. □

The next result shows that if f is bicritical but not a power map, then the group $\mathrm{Deck}_\infty(f) = \bigcup_{k=1}^{\infty} \mathrm{Deck}(f^k)$ studied by Pakovich is [6] is equal to $\mathrm{Deck}(f^3)$.

Proposition 5.4. *Let f be a bicritical rational map which is not a power map. Then $\mathrm{Deck}(f^k) = \mathrm{Deck}(f^3)$ for all $k \geq 3$.*

Proof. If $\mathrm{Deck}(f) = \mathrm{Deck}(f^2)$ or $\mathrm{Deck}(f^2) = \mathrm{Deck}(f^3)$, then the result holds by Lemma 4.4. Thus we may assume that $\mathrm{Deck}(f) \subsetneq \mathrm{Deck}(f^2) \subsetneq \mathrm{Deck}(f^3)$. Since $\mathrm{Deck}(f) \subsetneq \mathrm{Deck}(f^2)$, it follows from Lemma 4.6 that, using the notation of Lemma 5.2, $\mu_1 \in \mathrm{Deck}^*(f)$. Similarly, since $\mathrm{Deck}(f^3) \neq \mathrm{Deck}(f^2)$, there exists $\mu \in \mathrm{Deck}^*(f^2)$ such that $\mu(\mathcal{V}_f) = \mathcal{V}_f$. Such a map must be either μ_2 or μ_3 from Lemma 5.2. But since $\mu_3 = \mu_1 \circ \mu_2$ and $\mu_2 = \mu_1 \circ \mu_3$, we see that $\mu_2 \in \mathrm{Deck}^*(f^2)$ if and only if $\mu_3 \in \mathrm{Deck}^*(f^2)$. However, this means that $\{\mathrm{id}, \mu_1, \mu_2, \mu_3\} \subseteq \mathrm{Deck}(f^2)$ and so $\mathrm{Deck}^*(f^3) \cap \{\mathrm{id}, \mu_1, \mu_2, \mu_3\} = \varnothing$. Thus $\mathrm{Deck}^*(f^3)$ does not contain a Möbius transformation μ such that $\mu(\mathcal{V}_f) = \mathcal{V}_f$. But then Lemma 4.6 implies $\mathrm{Deck}(f^4) = \mathrm{Deck}(f^3)$, and so by Lemma 4.4 we have $\mathrm{Deck}(f^k) = \mathrm{Deck}(f^3)$ for all $k \geq 3$. □

6. Proofs of the Main Theorems

We are now ready to prove our main theorems.

6.1. **Proof of Theorem A.**

Proof of Theorem A. It is clear that if f is a power map then $\mathrm{Deck}(f^k) \cong \mathbb{Z}_{d^k}$. Now suppose f is not a power map, so that $|\mathcal{C}_f \cup \mathcal{V}_f| > 2$. If $|\mathcal{C}_f \cup \mathcal{V}_f| = 3$, then Lemma 5.1 asserts that $\mathrm{Deck}(f^k) \cong \mathbb{Z}_d$ for all k. If $|\mathcal{C}_f \cup \mathcal{V}_f| = 4$ then we note that since d is odd, $\mathrm{Deck}(f) \cong \mathbb{Z}_d$ cannot contain an element of order 2. But this means none of the elements μ_i, $i = 1, 2, 3$ from Lemma 5.2 can belong to $\mathrm{Deck}(f)$, and so by Lemma 4.6 we have $\mathrm{Deck}(f^2) = \mathrm{Deck}(f) \cong \mathbb{Z}_d$. Thus by Lemma 4.4, we have $\mathrm{Deck}(f^k) = \mathrm{Deck}(f) \cong \mathbb{Z}_d$ for all $k \geq 1$. □

6.2. **Proof of Theorem B.**

Proof of Theorem B. Again, we note that if f is a power map, then $\mathrm{Deck}(f^k) \cong \mathbb{Z}_{d^k}$ for all k. If f is not a power map, then by Lemmas 4.4 and 5.3, then either $\mathrm{Deck}(f^2) \cong D_{2d}$ or $\mathrm{Deck}(f^k) \cong \mathbb{Z}_d$ for all $k \geq 1$.

If $\mathrm{Deck}(f^2) \cong D_{2d}$, then by Theorem 2.4 and Lemma 4.8, the only possibilities for $\mathrm{Deck}(f^3)$ (up to isomorphism) are D_{4d} or D_{2d}. But by Proposition 5.4, the group $\mathrm{Deck}(f^k)$ cannot be larger than $\mathrm{Deck}(f^3)$, and this completes the proof. □

We conclude by showing that D_{2d} and D_{4d} are actually realized as $\mathrm{Deck}(f^k)$ for some bicritical rational map f of even degree d.

Proposition 6.1. *Let $d \geq 2$ be even.*

(1) *If $f(z) = \frac{z^d - a}{z^d + a}$ for some $a \neq 0$ then* $\mathrm{Deck}(f^2) \cong D_{2d}$.
(2) *If $g(z) = \frac{z^d - 1}{z^d + 1}$ then* $\mathrm{Deck}(g^3) \cong D_{4d}$.

Proof. As with Example 4.7, one could compute the groups $\mathrm{Deck}(f^2)$ and $\mathrm{Deck}(g^3)$ by hand, but we instead use some of our previously obtained results. We first note that since $\mathcal{C}_f = \mathcal{C}_g = \{0, \infty\}$ and d is even, the involution $\mu(z) = -z$ belongs to $\mathrm{Deck}(f)$ and $\mathrm{Deck}(g)$.

(1) It is clear that $\mathcal{V}_f = \{-1, 1\}$, and since $\mu(\mathcal{V}_f) = \mathcal{V}_f$, it follows from Lemmas 4.6 and 5.3 that $\operatorname{Deck}(f^2) \cong D_{2d}$.
(2) Let $\phi(z) = \frac{1}{z}$. Then a simple calculation yields

$$g \circ \phi(z) = -\frac{z^d - 1}{z^d + 1} = \mu \circ g(z).$$

Since from the first part we know $\mu \in \operatorname{Deck}(g)$, we see that by Lemma 4.3 we must have $\phi \in \operatorname{Deck}(g^2)$. Furthermore, it is clear that $\phi(\mathcal{V}_g) = \mathcal{V}_g = \{-1, 1\}$ and so $\operatorname{Deck}^*(g^3) \neq \varnothing$ by Lemma 4.6. But then $|\operatorname{Deck}(g^3)| = 2|\operatorname{Deck}(g^2)|$ by Lemma 4.8 and so $\operatorname{Deck}(g^3) \cong D_{4d}$. $\square$

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