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Constructing rational maps with cluster points using the mating operation

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Abstract

In this article, we show that all admissible rational maps with fixed or period 2 cluster cycles can be constructed by the mating of polynomials. We also investigate the polynomials that make up the matings that construct these rational maps. In the one-cluster case, one of the polynomials must be an n-rabbit and in the two-cluster case, one of the maps must be either f, a 'double rabbit', or g, a secondary map that lies in the wake of the double rabbit f. There is also a very simple combinatorial way of classifying the maps which must partner the aforementioned polynomials to create rational maps with cluster cycles. Finally, we also investigate the multiplicities of the shared matings arising from the matings in the paper.

1. *Introduction*

The study of holomorphic dynamical systems was first given undertaken seriously by Fatou [**11**, **12**] and Julia [**13**] in the early part of the 20th century. After laying relatively dormant for a number of years, the subject was given a new lease of life due to the improvements in technology which allowed mathematicians to view the various sets under investigation. In the mid 1980s, Hubbard and Douady produced their now famous 'Orsay lecture notes' [**6**, **7**], and since then the subject has grown from strength to strength.

This paper is a partner to [**22**], in which it was shown that, in certain cases, a very simple set of combinatorial data can classify (in the sense of Thurston equivalence) a rational map with a periodic cluster cycle. Like its partner paper, this paper is made up of results found in the author's PhD thesis [**21**]. Essentially, the former paper showed the uniqueness part of the classification of maps with cluster cycles, whereas the current paper is concerned with the existence part. We will show that all realizable combinatorial data for rational maps can be realized by matings of polynomials and so, when combined with the results of Sharland [**22**], we will show that all maps with cluster cycles are matings.

1.1. *Definitions*

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map on the Riemann sphere, of degree at least 2. The Julia set $J(f)$ will be the closure of the set of repelling periodic points of f, and the Fatou set is the set $F(f) = \widehat{\mathbb{C}} \setminus J(f)$. The connected components of $F(f)$ are called Fatou components. In the case where the critical orbits are periodic, the immediate basins of the (super)attracting orbit are called critical orbit Fatou components. We concern ourselves mainly with bicritical rational maps with marked critical points and such that the two critical points belong to the attracting basins of two disjoint periodic orbits of the same period. If f is a polynomial, then we define the filled Julia set $K(f)$ to be the set of points that are not attracted by the superattracting fixed point at infinity. It is well known that $\partial K(f) = J(f)$.

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Let F be a rational map for which every critical point is (pre-)periodic and suppose $U \subset F(f)$ is a Fatou component of period p. Then there exists an isomorphism $\phi_U : U \to \mathbb{D}$ such that $\phi_U \circ f^{\circ p} \circ \phi_U^{-1} : \mathbb{D} \to \mathbb{D}$ is the map $z \mapsto z^d$ for some d (see, for example, [**6**, **7**]). We call the set

$$\phi_U^{-1}(\{r\, e^{2\pi\theta} \,:\, 0 \leqslant r < 1\})$$

the *internal ray of angle* θ of the component U. If $\lim_{r\to 1}(\phi_U^{-1}(\{r\, e^{2\pi i\theta} \,:\, 0 \leqslant r < 1\}))$ exists, then we say that this limit is the endpoint of the internal ray.

Definition 1.1. Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a bicritical rational map. Then a cluster point for F is a point in $J(F)$, which is the endpoint of the angle 0 internal rays of at least one critical orbit Fatou component from each of the two critical cycles. We will define a cluster to be the union of the cluster point and the Fatou components meeting at it. The period of the cluster will be the period of the cluster point. The star of a cluster will be the union of the cluster point and the associated 0 internal rays, including the points on the critical orbit.

A standard example of a (fixed) cluster cycle is the rational map formed by the mating of the rabbit with the airplane; see Figure 1. The figure also gives examples of the terms in Definition 1.1. Note that, by our restriction to bicritical maps, the rational maps in this paper can have at most one cluster cycle.

We shall define the critical displacement of a rational map with either a fixed cluster point or a period 2 cluster cycle. Informally, the critical displacement will give a (combinatorial) measure as to how far apart the two critical orbits are in the clusters. It will be clear to the reader that the definitions of critical displacement are dependent on the labelling of the critical points of the map. In actual fact, we require a different version of the definition depending on whether we are studying the fixed case or the period 2 case. The reason for this is due to the result of the following lemma, which was proved in [**22**].

Lemma 1.1. *There does not exist a rational map F with a period 2 cluster cycle such that the critical points are in the same cluster.*

Definition 1.2. Let F be a rational map with a fixed cluster point. Label the endpoints of the star as follows. Let e_0 be the first critical point, and label the remaining arms in anticlockwise order by $e_1, e_2, \ldots, e_{2n-1}$. Then the second critical point is one of the e_j, and we call j the critical displacement of the cluster of F. We denote the critical displacement by δ.

We will sometimes use the fact that the critical displacement can equally well be calculated as the combinatorial distance between the critical values, as opposed to the critical points. Also, note that if one choice of marking for the critical points gives $\delta = k$, then the alternative marking will give a critical displacement of $\delta = -k$. Clearly, in light of Lemma 1.1, any attempt to use the above definition to define the critical displacement in the period 2 case is impossible, hence we give the following definition for the period 2 case.

Definition 1.3. Let F be a rational map with a period 2 cluster cycle. Choose one of the critical points to be c_1, and label the cluster containing it to be $\mathcal{C}_1$. Then (by Lemma 1.1) the other critical point c_2 is in the second cluster $\mathcal{C}_2$. We define the critical displacement δ as follows. Label the arms in the star of $\mathcal{C}_1$, starting with the arm with endpoint c_1, in anticlockwise order $\ell_0, \ell_1, \ldots, \ell_{2n-1}$. Then $F(c_2)$ is the endpoint of one of the ℓ_k. This integer k is the critical displacement, which we again denote by δ.

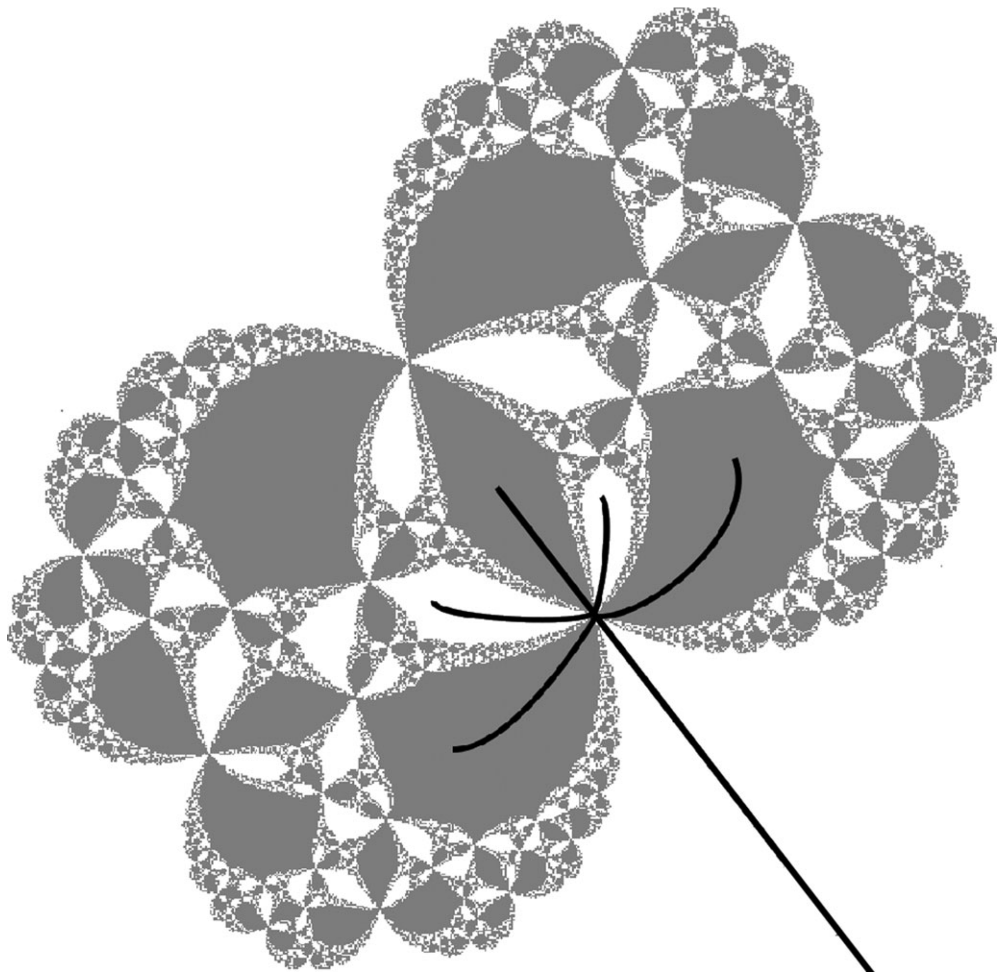

FIGURE 1. *A rational function with a fixed cluster point. This map is formed by the mating of the rabbit with the airplane. The black lines form the star of the cluster, with the central point being the cluster point itself. The cluster is formed by the union of the cluster point with the Fatou components containing the branches of the star.*

For this definition, we shall later be using the fact that the critical displacement is also equal to the combinatorial distance between $F(c_1)$ and $F^{\circ 2}(c_2)$. In contrast to the fixed case, there is not a neat symmetry to the two possible critical displacements that a rational map can have. We shall discuss this in further detail in Section 4.

DEFINITION 1.4. Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a bicritical rational map and let c be a cluster point of period n of F. Then the combinatorial rotation number is defined as follows. The first return map, $F^{\circ n}$, maps the star of the cluster, X_F to itself. Label the arms of the star (the 0 internal rays) which belong to one of the critical orbits (it does not matter which) cyclically in anticlockwise order by $\ell_0, \ell_1, \ldots, \ell_{q-1}$ (the initial choice of ℓ_0 is not important). Then, for each k, there exists p such that $F^{\circ n}$ maps ℓ_k to ℓ_{k+p}, subscripts taken modulo q. We then say that the combinatorial rotation number is $\rho = \rho(F) = p/q$.

We now define the combinatorial data of a rational map with a cluster cycle to be the pair (ρ, δ), where ρ is the combinatorial rotation number and δ is the critical displacement. As shown in [22], the combinatorial data of a map is enough to classify it (in the sense of Thurston) in the fixed case for all degrees and the quadratic period 2 case.

1.2. *Matings*

We will be constructing rational maps using matings. The mating of polynomials was first mentioned by Douady [5]. Informally, the construction allows us to take two complex polynomials f and g (along with their filled Julia sets $K(f)$ and $K(g)$) and paste them together to construct a branched cover of the sphere. We will informally consider the mating operation to be a map from the ordered pairs (f, g) of polynomials to the space of branched covers of the sphere. For further reading on this subject, see [15, 23, 25, 26].

Let f and g be two monic degree d polynomials defined on $\mathbb{C}$. In this paper, f and g will be unicritical but this is not needed in general. We define $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2\pi i t} : t \in \mathbb{R}/\mathbb{Z}\}$, the complex plane with the circle at infinity. We then extend the two polynomials to the circle at infinity by defining

$$f(\infty \cdot e^{2\pi i t}) = \infty \cdot e^{2d\pi i t} \quad \text{and} \quad g(\infty \cdot e^{2\pi i t}) = \infty \cdot e^{2d\pi i t}.$$

Label this extended dynamical plane of f by $\widetilde{\mathbb{C}}_f$ and the extended dynamical plane of g by $\widetilde{\mathbb{C}}_g$. We then create a topological sphere by gluing the two extended planes together along the circle at infinity. More formally, we define

$$S^2_{f,g} = \widetilde{\mathbb{C}}_f \uplus \widetilde{\mathbb{C}}_g / \sim$$

where $\sim$ is the relation that identifies the point $\infty \cdot e^{2\pi i t} \in \widetilde{\mathbb{C}}_f$ with the point $\infty \cdot e^{-2\pi i t} \in \widetilde{\mathbb{C}}_g$. We now define a new map, the formal mating $f \uplus g$ (Wittner called this the *non-intimate mating* in [26]), on this new space $S^2_{f,g}$ by defining

$$f \uplus g|_{\widetilde{\mathbb{C}}_f} = f \quad \text{and} \quad f \uplus g|_{\widetilde{\mathbb{C}}_g} = g.$$

We now wish to define an alternative type of mating, called the *topological mating*. This was essentially the form of mating that Douady introduced in [5]. First, we require a brief discussion on the theory of external rays. Suppose that the (filled) Julia set of the degree $d \geqslant 2$ monic polynomial $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is connected. Recall that, by Böttcher's theorem, there is a conformal isomorphism

$$\phi = \phi_f : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}} \setminus K(f),$$

which can be chosen so that it conjugates $z \mapsto z^d$ on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with the map f on $\widehat{\mathbb{C}} \setminus K(f)$.

Definition 1.5. Consider the radial line $r_t = \{r \exp 2\pi i t : r > 1\} \subset \mathbb{C} \setminus \mathbb{D}$. Then we call the set

$$R_f(t) = \phi_f(r_t)$$

the external ray of angle t.

If $K(f)$ (equivalently $J(f)$) is locally connected, then we can define the Carathéodory semiconjugacy $\gamma : \mathbb{R}/\mathbb{Z} \to J(f)$ such that

$$\gamma(t) = \gamma_f(t) = \lim_{r \to 1} \phi_f(r \exp(2\pi i t)).$$

The point $\gamma_f(t) \in K(f)$ is called the landing point of the external ray $R_f(t)$. Before moving on, we introduce some terminology. Given a polynomial of the form $f_c(z) = z^d + c$ with a locally connected Julia set, it is easy to see that the points $\beta_k = \gamma(k/(d-1))$ must be fixed. We call these the β-fixed points of f_c. There exists at most one other fixed point, which we shall call the

α-fixed point of f_c. We can also construct parameter rays analogously in the parameter plane. Let Φ be the (unique) Riemann map which gives a conformal isomorphism between $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\widehat{\mathbb{C}} \setminus \mathcal{M}_d$, with $\Phi(\infty) = \infty$ and such that Φ is asymptotic to the identity at infinity. Then the parameter ray of angle θ is the set

$$R_{\mathcal{M}_d}(t) = \Phi(\{r\exp 2\pi it : r > 1\}).$$

The notion of landing of rays is defined similarly to the case with external rays.

We now can define the topological mating of two monic degree d polynomials f_1 and f_2, assuming that they have locally connected Julia sets. We first define the ray-equivalence relation $\sim$ on $S^2_{f,g}$. The equivalence relation $\sim_f$ on $\widehat{\mathbb{C}}_f$ is generated by $x \sim_f y$ if and only if $x, y \in \bar{R}_f(t)$ for some t. Note that the closure of the external ray contains both the landing point and the point on the circle at infinity. Define a similar equivalence relation on $\widehat{\mathbb{C}}_g$. Then the equivalence relation $\sim$ will be generated by $\sim_f$ on $\widehat{\mathbb{C}}_f$ and $\sim_g$ on $\widehat{\mathbb{C}}_g$. We denote the equivalence class of x under this relation by $[x]$.

Denote the Carathéodory semiconjugacy derived from f_j by γ_j. We see that the ray-equivalence relation restricts to an equivalence relation $\sim'$ on the disjoint union of $K(f_1)$ and $K(f_2)$ by

$$\gamma_1(t) \sim' \gamma_2(-t) \quad \text{for each } t \in \mathbb{R}/\mathbb{Z}.$$

We define $K(f_1) \perp\!\!\!\perp K(f_2)$ to be the quotient topological space $S^2_{f,g}/\sim$, where every equivalence class is identified to a point. Making use of the fact that $\gamma_j(dt) = f_j(\gamma(t))$, we can piece together $f_1|_{K(f_1)}$ and $f_2|_{K(f_2)}$ to form a continuous map which we call $f_1 \perp\!\!\!\perp f_2$, the topological mating of f_1 and f_2. In nice cases, this quotient space $K_1 \perp\!\!\!\perp K_2$ is a topological sphere. We can think of the topological mating as the formal mating, where the external rays have been 'drawn tight' using the ray-equivalence relation. We remark that, in this paper, we will only be considering the mating of (monic unicritical) hyperbolic polynomials of degree $d \geqslant 2$. By the results of Pilgrim [**16**], this means that the Julia sets will be locally connected and so the topological mating will be well defined. Now suppose that we have constructed the topological mating $f_1 \perp\!\!\!\perp f_2$. We say that a rational map F is the geometric mating of the two monic polynomials f_1 and f_2 if there exists a topological conjugacy h that is orientation preserving and holomorphic on $\overset{\circ}{K}_1$ and $\overset{\circ}{K}_2$ satisfying

$$h \circ (f_1 \perp\!\!\!\perp f_2) = F \circ h.$$

In this case, we shall write $F \cong f_1 \perp\!\!\!\perp f_2$. It is possible that a rational map F may arise as a (geometric) mating in more than one way. Such a situation will be referred to as a shared mating.

1.3. *Thurston equivalence*

We now discuss how the mating can be used to define a rational map on the Riemann sphere. As it stands, the formal mating is a branched covering of the sphere. To provide a conformal structure to the topological sphere, we need to invoke the results of Thurston on the classification of branched covers. Recall that a branched covering $F : S^2 \to S^2$ is called post-critically finite if the post-critical set

$$P_F = \bigcup_{n>0} F^{\circ n}(\{z : z \text{ is a critical point of } F\})$$

is finite.

DEFINITION 1.6. Let $F, G : S^2 \to S^2$ be post-critically finite orientation-preserving branched self-coverings with labelled critical points. An *equivalence* is given by a pair of orientation-preserving homeomorphisms (ϕ_0, ϕ_1) such that the following conditions are satisfied:

(i) $\phi_0|_{P_F} = \phi_1|_{P_F}$;
(ii) the following diagram commutes:

$$\begin{array}{ccc} (S^2, P_F) & \xrightarrow{\phi_0} & (S^2, P_G) \\ \downarrow F & & \downarrow G \\ (S^2, P_F) & \xrightarrow[\phi_1]{} & (S^2, P_G) \end{array}$$

(iii) ϕ_0 and ϕ_1 are isotopic via homeomorphisms ϕ_t, $t \in [0, 1]$ satisfying $\phi_0|_{P_F} = \phi_t|_{P_F} = \phi_1|_{P_F}$ for each $t \in [0, 1]$.

We say that F and G are (Thurston) *equivalent* if there exists an equivalence as defined above. In this case, we shall write $F \cong G$.

Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ be a collection of curves in S^2. If the $\gamma_i \in \Gamma$ are simple, closed, non-peripheral, disjoint and non-homotopic relative to P_F, then we say that Γ is a multicurve. We say that the multicurve is F-stable if, for any $\gamma_i \in \Gamma$, all the non-peripheral components of $F^{-1}(\gamma_i)$ are homotopic rel $S^2 \setminus P_F$ to elements of Γ. Given an F-stable multicurve, we can define a non-negative matrix $F_\Gamma = (f_{ij})_{n\times n}$ in the following natural way. For each i, j, let $\gamma_{i,j,\alpha}$ be the components (these are all simple, closed curves) of $F^{-1}(\gamma_j)$ that are homotopic to γ_i in $S^2 \setminus P_F$. Now define

$$F_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{\deg F|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j} \gamma_i,$$

where deg denotes the degree of the map. By standard results on non-negative matrices (see [**20**]), this matrix (f_{ij}) will have a leading non-negative eigenvalue λ. We write $\lambda(\Gamma)$ for the leading eigenvalue associated with the multicurve Γ.

Definition 1.7. The multicurve Γ is called a Thurston obstruction if $\lambda(\Gamma) \geqslant 1$.

Theorem 1.2 (Thurston's theorem). *A post-critically finite branched covering $F : S^2 \to S^2$ of degree $d \geqslant 2$ with hyperbolic orbifold is equivalent to a rational map R on the Riemann sphere if and only if F has no Thurston obstructions. Furthermore, any equivalence between two rational maps is realized by a pair (μ, μ), where μ is a Möbius transformation.*

The requirement for F to have a hyperbolic orbifold in the above theorem will not concern us in this paper, since all of our branched coverings will have hyperbolic orbifolds. Indeed, if F is a branched covering with non-hyperbolic orbifold, then it must have $P_F \leqslant 4$ and the pre-image of the post-critical set must be contained in the union of the post-critical set and the set of critical points of F. The interested reader should refer to [**8**] for a proof and discussion of the above result. Although Thurston's theorem is a very strong result, it has some problems in applications, namely it is difficult to check in general for Thurston obstructions.

Definition 1.8. A multicurve $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is a Levy cycle if, for each $i = 1, \dots, n$, the curve γ_{i-1} (or γ_n if $i = 1$) is homotopic to some component γ'_i of $F^{-1}(\gamma_i)$ (rel P_F) and the map $F : \gamma'_i \to \gamma_i$ is a homeomorphism. A Levy cycle is a good Levy cycle if the connected components of $S^2 \setminus \bigcup_{i=1}^{n} \gamma_i$ are $D_1, \ldots, D_m, C$, with the D_j all being discs. When $n = 1$, we have $C = \varnothing$ and $F|_{\gamma_1'} : \gamma_1' \to \gamma_1$ reverses the orientation. For $n > 1$, there exists a component C' of $F^{-1}(C)$ that is isotopic to C and such that $F|_{C'} : C' \to C$ is a homeomorphism.

A discussion of different types of Levy cycles can be found in [**23**]. The following result, the culmination of work by Rees, Shishikura and Tan, greatly simplifies the search for Thurston obstructions in the bicritical case.

PROPOSITION 1.3 [**24**]. *In the bicritical case, F has a Levy cycle if and only if it has a Thurston obstruction.*

A related result in [**25**] is the following theorem.

THEOREM 1.4. *Let f_1, f_2 be monic unicritical polynomials of the form $f_i(z) = z^d + c_i$ and with α-fixed points labelled as α_1 and α_2, respectively. Then the following are equivalent:*

(i) *F has a (good) Levy cycle $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$;*
(ii) *F has a ray-equivalence class τ containing closed loops and two distinct fixed points;*
(iii) $[\alpha_1] = [\alpha_2]$;
(iv) *f_1 and f_2 are in conjugate limbs of the degree d multibrot set.*

This result significantly reduces the work required to check whether the mating of two unicritical monic polynomials is obstructed, since it suffices by condition (iii) to check that the ray classes of $[\alpha_1]$ and $[\alpha_2]$ are disjoint. Note that the previous two results show that, in particular, a mating is obstructed if and only if f_1 and f_2 belong to conjugate limbs of $\mathcal{M}_d$. We finish with a slight generalization of Theorem 1.4, which we shall use in the next section.

LEMMA 1.5 [**23**]. *Let F be a mating. Let $[x]$ be a periodic ray class such that $[x]$ contains a closed loop. Then each boundary curve of a tubular neighbourhood of $[x]$ generates a Levy cycle.*

By a result of Rees [**17**], in the hyperbolic post-critically finite case, if the formal mating of f_1 and f_2 is Thurston equivalent to the rational map F, then F will be the geometric mating of f_1 and f_2. Hence, we can use any of the different notions of mating when discussing a mating, without worrying that a different choice will affect the Thurston class.

1.3.1. *A combinatorial view of matings.* In this section, we discuss the (periodic) ray classes that occur in the formal matings, which are then collapsed to points in the topological mating. This discussion is extremely natural and will allow us to focus on the important ray classes (those which become the cluster points) in the later sections.

LEMMA 1.6. *Let $F = f_1 \uplus f_2$ be the formal mating of two hyperbolic polynomials which has no Thurston obstruction. Let $z_0, F(z_0) = z_1, \ldots, F^{\circ(n-1)} = z_{n-1}$ be a period n orbit of f_1 that is contained in $J(f_1)$ and has combinatorial rotation number different from 0. Then the periodic ray classes $[z_0], [z_1], \ldots, [z_{n-1}]$ are pairwise disjoint.*

Proof. Suppose that there exists k with $[z_0] = [z_k]$. Then there exists a path through external rays γ from z_0 to z_k. The map $F^{\circ n}$ will take z_0 to z_0 and z_k to z_k and takes the path γ to some path γ', which is a path from z_0 to z_k. However, γ' is not equal to γ, since the first return map to z_0 and z_k will permute the external rays landing there. Hence, the union $\gamma \cup \gamma'$ contains a loop and so the mating will be obstructed by Lemma 1.5. □

LEMMA 1.7. *If the mating of two hyperbolic polynomials is not obstructed, then each periodic ray class contains at most one periodic branch point with non-zero combinatorial rotation number.*

Proof. Let w_0 and z_0 be two periodic branch points with non-zero combinatorial rotation number, such that $[w_0] = [z_0]$. We will show that the periods of z_0 and w_0 are equal. Let the period of z_0 be n. Then the map $F^{\circ n}$ maps z_0 to itself, the periodic ray class $[z_0]$ to itself and w_0 to $w_n = F^{\circ n}(w_0)$. Then we must have $[w_0] = [z_0] = [w_n]$ and so $w_0 = w_n$ by Lemma 1.6. Hence, the period of w_0 is divisible by n. An analogous argument shows that the period of w_0 divides the period of z_0, and so the periods are the same. Denote this common period by n.

Now let γ be a path through external rays from w_0 to z_0. Since none of the rays meet a pre-critical point (since the maps f_1 and f_2 are hyperbolic), the nth iterate of γ, which we call γ', will also be a path from w_0 to z_0. Since the external rays at w_0 and z_0 are permuted under the first return map, then $\gamma \neq \gamma'$, and so the curve $\gamma \cup \gamma'$ contains a loop, whence the mating is obstructed. □

The importance of Lemma 1.7 is that it will tell us the nature of the maps that make up the matings that produce maps with cluster cycles. Intuitively, it seems clear that to create a rational map with a cluster cycle of period p and rotation number ρ, it is necessary for one of the maps to have a periodic cycle of period p that has combinatorial rotation number ρ, so that this cycle for the polynomial will become the cluster point cycle. We will show that this intuition is correct, at least in the cases for periods 1 and 2 (though analogous proofs exist for all periods). The above Lemma 1.7, on the other hand, sheds some light on the behaviour of the other polynomial in the mating. In short, it will allow us to say that the second polynomial must belong to a satellite component in the multibrot set.

Note that if we have a periodic ray class that contains a point p with combinatorial rotation number not equal to 0, then the ray class must contain a sort of rotational symmetry: each global arm at the point p (a global arm is a component of the complement to p in the ray class) is homeomorphic to each of the other global arms. Furthermore, if the ray class is periodic, then it cannot contain a strictly pre-periodic point, and all points in the ray class have period dividing the period of the ray class.

LEMMA 1.8. *Suppose that a periodic ray class contains a point p that has combinatorial rotation number different from 0. Then, given any periodic orbit $\mathcal{O} = \mathcal{O}(z_0) = \{z_0, z_1, \ldots, z_{n-1}\}$, the intersection between any global arm at p and $\mathcal{O}$ contains at most one point.*

Proof. Clearly, if $[p] \neq [z_i]$ for any i, then the intersection of $\mathcal{O}(z)$ with any global arm will be empty. So assume without loss of generality that $[p] \cap [z_0] \neq \varnothing$. Let ℓ be a global arm at p in the periodic ray class. There exists some k so that $F^{\circ k}(\ell) = \ell$. If $\ell \cap \mathcal{O}$ is empty, then ℓ also contains no pre-images of points in $\mathcal{O}$, since otherwise some forward image of ℓ will contain a point in $\mathcal{O}$, and so $\ell \cap \mathcal{O}$ would be non-empty, which is a contradiction. Since ℓ contains no pre-images of points in $\mathcal{O}$, no forward iterate of ℓ can contain points of $\mathcal{O}$. Since global arms are mapped homeomorphically onto global arms, this means that all the global arms have empty intersection with the orbit $\mathcal{O}$. Hence, every global arm at p must contain at least one point of $\mathcal{O}$.

Suppose that the global arm ℓ contains $r > 1$ elements of $\mathcal{O}$, $z_0, \ldots, z_{r-1}$. Under the first return map to ℓ, these elements must be permuted, since the orbit is periodic. Without loss of generality, we can assume that the first element of $\ell \cap \mathcal{O}$ on the global arm (in terms of distance from p) is z_0. Then the second point (again, in terms of distance in the tree from p) in $\ell \cap \mathcal{O}$ is $F^{\circ k}(z_0) = z_k$ for some k. Let γ be the sub-arm from p to z_0. Then $\gamma' = F^{\circ k}(\gamma)$ is contained in ℓ and will be a path from p to z_k since the global arm ℓ maps homeomorphically onto itself under the return maps (since the map F is a homeomorphism on the graphs of the ray-equivalence classes). In particular, $z_0 \in \gamma'$. However, this means that there has to be a pre-image of z_0 in the path γ. However, by construction, there are no points in the orbit of z_0 in the interior of γ and there also do not exist any pre-periodic points in γ. This contradiction means that the global arm ℓ must contain at most one point in $\mathcal{O}$. □

1.4. *Main results*

The focus of this paper will be the proof of the following two theorems. The definitions of angular rotation number and 2-angular rotation number can be found in Definitions 2.1 and 3.1, respectively. Note that Main Theorem 2 is only concerned with the quadratic case.

MAIN THEOREM 1. *Suppose that F is a bicritical rational map with a fixed cluster point and the combinatorial rotation number is p/q. Then the following conditions hold*:

(i) *F is the mating of a q-rabbit (whose α-fixed point has combinatorial rotation number p/q) and another map h;*
(ii) *the map h has an associated angle with angular rotation number $(q - p)/q$.*

In his thesis, Wittner [26] showed that the mating of Douady's rabbit with the airplane was equivalent to the mating of the airplane to Douady's rabbit (Figure 1) using the criterion that any rational map that is a mating should exhibit an equator (see, for example, [23]). He made use of a combinatorial move known as the 'Wittner flip'; this method has been further developed in the work of Rees [17] and Exall [10], among others. As a generalization of this, an observation of Epstein showed that, for quadratic maps, the rational map with combinatorial data $(1/q, 1)$ could be obtained in two different ways. He showed that the mating of the $1/q$-rabbit[†] with the unique period q component in the wake of the $(q - 1)/q$-rabbit is equivalent to the mating of the period q airplane[‡] with the $1/q$-rabbit. Main Theorem 1 generalizes these results by showing that all (degree d) bicritical rational maps with fixed cluster points can be obtained by matings.

MAIN THEOREM 2. *Suppose that F is a quadratic rational map with a period 2 cluster point and the combinatorial rotation number is p/q. Then the following conditions hold.*

(i) *the map F is a mating;*
(ii) *precisely one of the maps in the mating belongs to the $1/2$-limb of the Mandelbrot set and this map is one of the two period $2q$ maps which belong to the p/q-sublimb of the period 2 component of the Mandelbrot set;*
(iii) *the other map has an associated angle with 2-angular rotation number $(q - p)/q$.*

[†]The q-rabbit whose α-fixed point has combinatorial rotation number $1/q$.

[‡]The map $f_c : z \mapsto z^2 + c$ with c being the smallest real number so that 0 has period q under f_c.

Rees observed a special case of Main Theorem 2 in the period 4 case. She showed that the mating of the 'double basilica' with 'kokopelli' was equivalent to the mating of the period 4 airplane with 'co-kokopelli'[§]. Main Theorem 2 and the results of Sharland [**22**] show that such equivalences can also be found for matings of polynomials of a higher period.

We also will prove the following fact about the multiplicity of the shared matings in the period 2 cluster case.

MAIN THEOREM 3. *In the period 2 cluster cycle case, each set of combinatorial data can be obtained in either two, three or four ways and all of these multiplicities are obtained by some shared mating.*

2. *Fixed cluster points*

Our goal in this section will be to prove Main Theorem 1. The results of this section are relatively elementary, but provide an interesting insight into the differences between the fixed case and the period 2 case. It will also allow us to introduce some terminology that will be of use in the period 2 cluster cycle case.

PROPOSITION 2.1. *Suppose that F is a bicritical rational map with a fixed cluster point, where both critical points have period n. Then the two critical point components cannot be adjacent. In other words, the critical displacement cannot be* 1 *or* $2n-1$.

Proof. We will assume, to obtain a contradiction, that U_0 is the Fatou component that immediately follows V_0 in the anticlockwise cyclic ordering of components around the cluster point c, and that U_0 and V_0 are the components containing the critical points of F. We will use the notation that $U_k = F^{\circ k}(U_0)$ and $V_k = F^{\circ k}(V_k)$ to label the critical orbit components. Let γ_1 be a curve that loops around v_1 and v_2, as in Figure 2, so that the only critical orbit Fatou components intersecting with γ_1 are U_1 and U_2. Now define $\gamma_n = F^{\circ(n-1)}(\gamma_1)$ for $n = 1, \ldots, n-1$. Then $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a Levy cycle. □

This result tells us that the two critical points cannot be in adjacent Fatou components in the cluster. We will actually show that this is the only restriction on the combinatorial data for a rational map with a fixed cluster point. We now prove the first part of our classification: that one of the maps must be a q-rabbit; that is, a map that belongs to a hyperbolic component that bifurcates from the (unique) period 1 component in the degree d multibrot set $\mathcal{M}_d$.

PROPOSITION 2.2. *Let f_1 and f_2 be monic unicritical polynomials with a period q superattracting orbit. Suppose $F \cong f_1 \perp\!\!\!\perp f_2$ is a rational map that has a fixed cluster point. Then either f_1 or f_2 is a q-rabbit.*

Proof. A degree d rational map has $d+1$ fixed points (counting multiplicity), and so we see that one of the α-fixed points, α_1 or α_2, must become the fixed cluster point in the rational map, since otherwise we would have $[\alpha_1] = [\alpha_2]$ and the mating would be obstructed, by Theorem 1.4. Without loss of generality, let the cluster point of F be $[\alpha_1]$.

[§]The names are due to Milnor. Furthermore, the period 4 airplane is sometimes referred to as the 'Airbus' or the 'worm'. The double basilica is the polynomial $f_c : z \mapsto z^2 + c$ for $c = -1.31070\ldots$, kokopelli is $c = (-0.15652\ldots + 1.03225\ldots i)$, co-kokopelli is $c = (-0.15652\ldots - 1.03225\ldots i)$ and the Airbus is $c = -1.94080\ldots$.

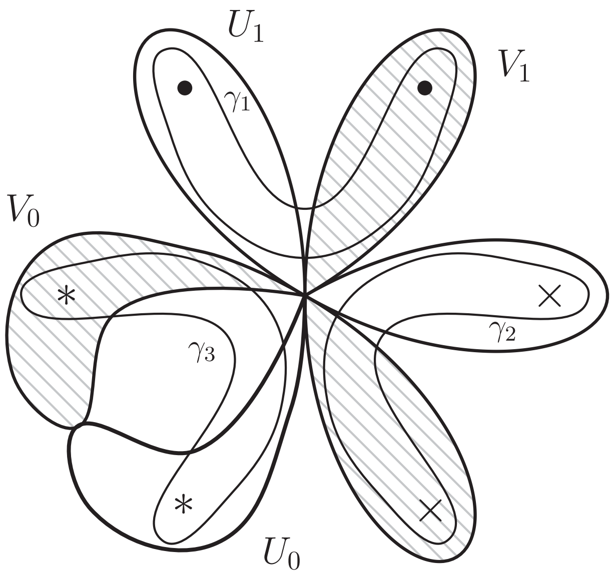

FIGURE 2. *The multicurve* Γ. *The asterisks represent critical points, the dots critical values and the* $\times$*s represent other elements of the post-critical set.*

We see that α_1 is the landing point of k external rays and these are permuted cyclically under iteration by the map f_1. Since the mating is not obstructed, $[\alpha_1]$ is a tree and the global arms of the tree are permuted cyclically and homeomorphically under $f_1 \uplus f_2$. Furthermore, because the map $f_1 \uplus f_2$ is a homeomorphism on this tree, each global arm contains at most one root point of a critical orbit Fatou component of f_2. Since there must be q root points of critical orbit Fatou components in this ray-equivalence class, there are q global arms at α_1 and so q external rays landing at α_1. But then f_1 has a period q superattracting cycle and q external rays landing on its α-fixed point, so it is a q-rabbit. □

Indeed, we can say slightly more. If the rational map has a fixed cluster point with combinatorial rotation number $\rho = p/q$, then the rabbit is actually the p/q-rabbit, the rabbit whose α-fixed point has rotation number p/q. This is easy to see by noting that the combinatorial rotation number at the α-fixed point is given by the ordering of the permutation of the external rays, and it is precisely these external rays that form the arms of the periodic ray class that become the cluster point.

2.1. *Properties of the non-rabbit*

To complete the classification of the maps that mate to give a rational function with a fixed cluster point, we need to study the properties of the map that partners the q-rabbit in the mating. Indeed, our classification of this complementary map requires us to take into account the ordering of the external rays that land at the α-fixed point of the rabbit. This classification will follow from the considerations found in [**1**, **4**]. In particular, we need to provide the definition of the rotation number of an angle.

DEFINITION 2.1. Let $\theta \in S^1$ be periodic of period q under the map $\sigma_d : t \mapsto dt$, so that $\theta = a/(d^q - 1)$ for some a. Label the angles $d^j\theta$, $j = 1, \dots, q$ cyclically by $\theta_1, \theta_2, \dots, \theta_n$ with

$\theta_1 = \theta$. Then we say that the angle θ has (angular) rotation number p/q if

$$d\theta_k = \theta_{k+p \bmod q}$$

for each k.

Note that if θ has rotation number p/q, then the angles $d\theta$, $d^2\theta, \ldots$ have rotation number p/q also. Hence, we can equally well talk of the orbit of angles having angular rotation number p/q.

Definition 2.2. Let f be a hyperbolic polynomial belonging to the degree d multibrot set $\mathcal{M}_d$. We say that the angle θ is associated with the polynomial f if the external ray of angle θ lands at a (not necessarily principal) root point of the critical value Fatou component of f.

The following is well known and gives an extremely useful link between the dynamical plane and the parameter plane; see, for example, [**14**].

Proposition 2.3. *The angle θ is associated with f if and only if the parameter ray of angle θ lands at a (not necessarily principal) root point of the hyperbolic component containing the map f.*

In what follows, we use the notation A_k to denote the arc $(k/(d-1), (k+1)/(d-1)) \subset S^1$.

Lemma 2.4. *Suppose $F = f \perp\!\!\!\perp h$ is a rational map with a fixed cluster point and f_1 is the p/q-rabbit. Then one of the angles associated with h has rotation number $(q-p)/q$. Moreover, in degree d, all the angles in the forward orbit of this angle lie in an arc A_j.*

Proof. All rays landing on the α-fixed point of f have angles θ_i (in cyclic order) with rotation number p/q and lie in an arc A_k for some $k \in \{0, 1, \ldots, d-2\}$. Since the only periodic biaccessible point on $J(f_1)$ is α_1 [**26**, Claim 10.1.1], the root points of the critical orbit Fatou components must be the landing points of the rays of angles $-\theta_i$. Each of these angles has rotation number $(q-p)/q$ and precisely one of them is the angle of the ray landing at a root point of the critical value Fatou component of h. Since it lands at a root point of the critical value component of h, it is one of the angles associated with h. The second statement simply follows from the fact that all the angles θ of the external rays landing at the α-fixed point of an n-rabbit lie in some arc A_k, and so the angles $-\theta$ must lie in the arc A_{d-k-2}. □

Proposition 2.5. *All (admissible) combinatorial data can be obtained by matings in (precisely) $2(d-1)$ ways.*

Proof. First, we shall construct a rational map $F \cong f \perp\!\!\!\perp h$ with combinatorial data (ρ, δ). We see that ρ is of the form p/q, so we take f to be the p/q-rabbit whose corresponding angles lie in the arc A_0. We now make a judicious choice for our complementary map h. Label the angles of the rays landing at the α-fixed point of f in cyclic order by $\theta_1, \ldots, \theta_q$, starting with θ_1 as the angle of the ray that lands anticlockwise from the critical value component of f. To get a critical displacement of $\delta = 2k-1$, we choose h to be the rational map corresponding to the angle $-\theta_k$. The map h does not lie in the conjugate limb to f in $\mathcal{M}_d$ and so the mating exists and the rational map has combinatorial data (ρ, δ).

To see that all combinatorial data is obtained with multiplicity $2(d-1)$, observe that the p/q-rabbit f that was chosen in the previous paragraph could equally well have been the one whose corresponding angles lay in the arc A_k, with $k \in \{0, \dots, d-2\}$. Then a similar argument allows us to find a complementary map h, which means $f \perp\!\!\!\perp h$ has combinatorial data $(p/q, \delta)$. Moreover, we can carry out the mating $h' \perp\!\!\!\perp f$, where h' is the map for which $f \perp\!\!\!\perp h'$ has critical displacement $-\delta$ and again this holds for any choice of the p/q-rabbit f. This gives us the $2(d-1)$ distinct ways of forming the mating.

To see that $2(d-1)$ is sharp, observe that, by results of Blokh, Malaugh, Mayer, Oversteegen and Parris [**1**, Theorem 2.8], there is precisely one orbit of angles in each arc A_k which has rotation number $(q-p)/q$[†]. These angles are precisely the angles $-\theta_1, \dots, -\theta_q$ above. Two of these (namely, $-\theta_1$ and $-\theta_q$) correspond to the $(q-p)/q$-rabbit and each of the others constructs a rational map with a distinct critical displacement when mated with the appropriate p/q-rabbit. So no other maps can be mated with a p/q-rabbit to create a rational map with a fixed cluster point. □

REMARK 2.1. The following can be deduced from the above proof and a simple counting argument. Suppose that h is a polynomial such that $f \perp\!\!\!\perp h$ has a fixed cluster cycle (for some choice of the p/q-rabbit f). Then h has a corresponding angle θ with angular rotation number p/q and whose orbit $\{\theta, d\theta, \dots, d^{q-1}\theta\}$ lies in an arc A_k.

2.2. *Proof of Main Theorem* 1

We now prove the first of our two main theorems. First we state a result from [**22**].

THEOREM 2.6 [**22**, Theorem A]. *Suppose that F and G are bicritical rational maps* (*with labelled critical points*) *with fixed cluster cycles with the same combinatorial data. Then F and G are equivalent in the sense of Thurston.*

Proof of Main Theorem 1. By Proposition 2.2, one of the maps must be an n-rabbit, f, and since combinatorial rotation numbers are preserved by matings and the fact that the α-fixed point of f becomes (under the mating) the cluster point, the combinatorial rotation number of the α-fixed point of f must be p/q as well. By Lemma 2.4, one of the angles associated with h must have rotation number $(q-p)/q$. Since all admissible combinatorial data can be obtained by matings (Proposition 2.5), then, by Theorem 2.6, all rational maps with fixed cluster points are matings. □

3. *Period* 2 *cluster cycles*

The simplicity of the results in the fixed cluster point case perhaps could lead the reader to believe that analogous results hold in the period 2 case. However, there is an increased level of complexity in the period 2 case. We shall only consider the quadratic case in this section: a brief discussion of the higher-degree case (and the difficulties in tackling it) is given after Theorem 3.1. We know from the previous section that if we wanted to construct a rational map with a fixed cluster point that has combinatorial rotation number ρ, then we need one of the

[†]In degree 2, this essentially states the well-known fact that there is a unique p/q-rabbit.

maps to be the ρ-rabbit. We do not get such an exact statement in the period 2 case. However, we can differentiate between the two polynomials: one of them must belong to the 1/2-limb.

THEOREM 3.1. *Let $F \cong f_1 \perp\!\!\!\perp f_2$ be a degree 2 rational map with a period 2 cycle of cluster points. Then precisely one of the maps f_1 or f_2 belongs to $\mathcal{M}_{1/2}$, the 1/2-limb of $\mathcal{M}$.*

Proof. It is clear that both f_1 and f_2 cannot belong to the 1/2-limb, since then the mating would be obstructed by Theorem 1.4. Hence, it only remains to show that not both of f_1 and f_2 can lie outside $\mathcal{M}_{1/2}$. So assume that f_1 and f_2 lie outside $\mathcal{M}_{1/2}$. If f_i is not in $\mathcal{M}_{1/2}$, then the external rays of angles 1/3 and 2/3 must land at distinct points. Since these angles have period 2 under angle doubling, they must land at points with period dividing 2. As $R_{1/3}$ and $R_{2/3}$ land at different points, these landing points must be a period 2 cycle.

Now note that, under mating, we have the identifications

$$\gamma_{f_1}(1/3) \sim \gamma_{f_2}(2/3) \quad \text{and} \quad \gamma_{f_2}(1/3) \sim \gamma_{f_1}(2/3),$$

and these points are not identified with each other, or any other points. In particular, these points cannot be cluster points. Since $f_i(\gamma_{f_i}(1/3)) = \gamma_{f_i}(2/3)$ and $f_i(\gamma_{f_i}(2/3)) = \gamma_{f_i}(1/3)$, these pairs form a period 2 cycle for the map $F \cong f_1 \perp\!\!\!\perp f_2$. However, since F already has a period 2 cycle (the cluster point cycle) by assumption, we see that this second period 2 cycle cannot exist, since a degree 2 rational map can only have one period 2 cycle. Hence, both of the maps cannot lie outside $\mathcal{M}_{1/2}$, and so precisely one of them lies in $\mathcal{M}_{1/2}$. □

We note that, similarly to the last section, we can separate the classification into an investigation of the map which is in $\mathcal{M}_{1/2}$ and the map which is not. A further comment is required on the restriction to degree 2. On the one hand, this restriction is motivated by the fact that Thurston equivalence is only possible with the given combinatorial data in the quadratic case. However, it turns out that even a generalization of Theorem 3.1 does not hold in the higher-degree case. In particular, although it remains true that one of the maps must belong to a 1/2-limb† in the degree d multibrot set $\mathcal{M}_d$, it is not true that precisely one of them has this property. Examples showing that this is no longer the case can be found in [**22**, Section 4] and also in the forthcoming paper [**9**]. It is hoped that a more detailed study of the higher-degree case will be the subject of future work.

3.1. *Properties of the map in $\mathcal{M}_{1/2}$*

For each $\rho \in \mathbb{Q}/\mathbb{Z}$, there are two maps in the 1/2-limb that have a period 2 orbit with combinatorial rotation number ρ. It turns out that either of these can be used to create a rational map with a period 2 cluster cycle using matings.

PROPOSITION 3.2. *Let f_1 and f_2 be quadratic polynomials with period $2q$ superattracting orbits. Suppose $F \cong f_1 \perp\!\!\!\perp f_2$ is a rational map that has a period 2 cluster cycle. Then one of f_1 or f_2 is either the tuning of the basilica by a q-rabbit (a 'double rabbit'), or the (unique) other period $2q$ component lying in the wake of this double rabbit.*

Proof. This proof is equivalent to showing that the map in $\mathcal{M}_{1/2}$ must belong to a p/q-sublimb of the period 2 component of $\mathcal{M}$. Note that maps belonging to this sublimb are precisely those that have a period 2 point with combinatorial rotation number p/q. If $f \in \mathcal{M}_{1/2}$, then

†The notion of the 1/2-limb is no longer unique in degrees greater than 2.

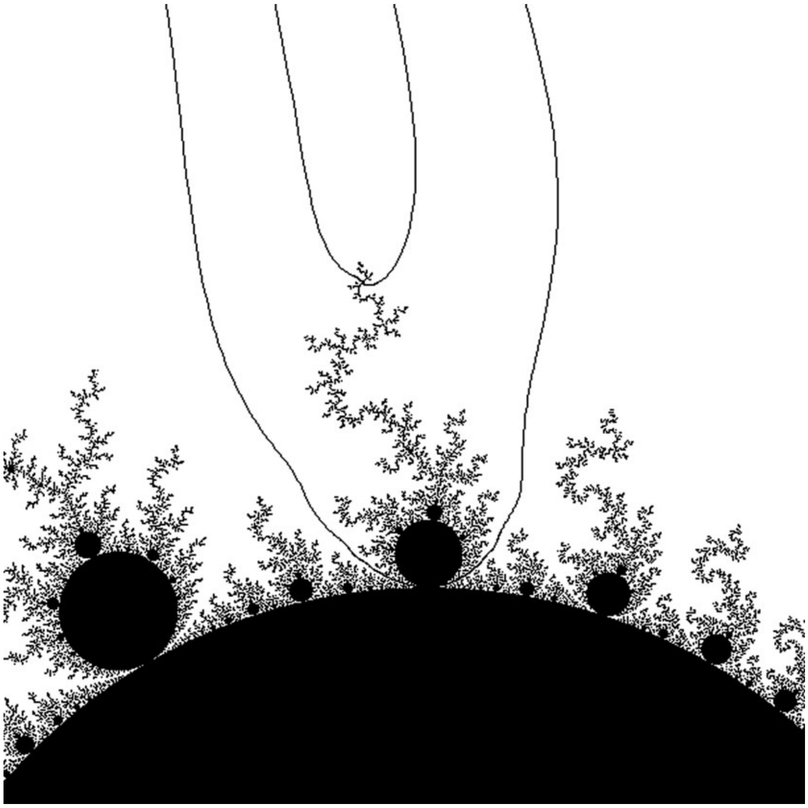

FIGURE 3. *The double rabbit component and secondary map component of period 8, rotation number* 1/4 *case.*

it has a period 2 cycle $\{p_1, p_2\}$. Under mating, the equivalence classes of these points must also have period 2, since otherwise we would have $[p_1] = [p_2]$, contradicting Lemma 1.7. Since a quadratic rational map has at most one period 2 cycle, the classes $[p_1]$ and $[p_2]$ must become the cluster cycle. It now follows from Lemma 1.8 that the combinatorial rotation number of this period 2 orbit will be p/q for some p, and the only polynomials that have a period $2q$ superattracting cycle and a period 2 point with combinatorial rotation number p/q are the two given in the statement of the proposition. □

It again follows easily that the combinatorial rotation number of the period 2 cycle is the same as the combinatorial rotation number of the period 2 cluster cycle in the resulting rational map. Figure 3 shows the position of these two maps in the period 8, rotation number 1/4 case. We remark for the moment that this classification is the best possible: it is not true in the period 2 cluster case that one of the maps must be the double rabbit. The matings with the double rabbit are the canonical examples, in that they behave similarly to the examples found in Section 2. We shall consider the matings with the secondary map in Subsection 3.4. We denote by $f_{p/q}$ (respectively, $g_{p/q}$) the double rabbit (respectively, secondary map) with a period 2 orbit with combinatorial rotation number p/q.

3.2. *Properties of the complementary map*

We now attempt to prove some analogous results to those in Subsection 2.1. We start with a well-known lemma whose proof, which makes use of results from [**14**], is omitted.

LEMMA 3.3. *Suppose that z is a biaccessible periodic point in $J(f_{p/q})$. Then z is either the α-fixed point or belongs to the period 2 orbit $\{p_1, p_2\}$.*

As with the previous section, our description of the complementary map (the map that does not belong to the 1/2-limb of the Mandelbrot set) will take place as a discussion of the associated angles. In fact, there are perhaps more descriptive ways of describing these maps (see [**21**] for a discussion of the combinatorial classification using internal addresses), but the description with associated angles is the more complete for the moment. We first require an analogue to Definition 2.1.

DEFINITION 3.1. Let $\theta \in S^1$ be periodic of period $2q$ under the map $\sigma_2 : t \mapsto 2t$ (so $\theta = a/(2^{2q} - 1)$ for some a). Consider the disjoint sets

$$\begin{aligned} A_0 &= A_0(\theta) = \{\theta, 2^2\theta, 2^4\theta, \ldots, 2^{2k}\theta, \ldots, 2^{2(q-1)}\theta\} \\ \text{and} \quad A_1 &= A_1(\theta) = \{2\theta, 2^3\theta, \ldots, 2^{2k+1}\theta, \ldots, 2^{2q-1}\theta\}. \end{aligned}$$

We say that θ has (admissible) (2-)angular rotation number p/q if the following conditions are satisfied:

(i) A_0 and A_1 lie in disjoint arcs of S^1;
(ii) the sets A_0 and A_1 have (angular) rotation number (in the sense of Definition 2.1) p/q under the map $\theta \mapsto 4\theta$.

Clearly, this is not a full generalization of Definition 2.1, since the first condition above introduces a restriction that was not used previously. There are orbits of angles that satisfy the second condition but not the first (see [**1**]), but we do not want to consider such orbits in this paper and so the restriction allows us to ignore them. For the moment, we will be assuming that the map in $\mathcal{M}_{1/2}$ is the double rabbit. As will be shown later, the set of complementary maps to the secondary map are a subset of the complementary maps to the double rabbit.

PROPOSITION 3.4. *Suppose $F \cong f_{p/q} \perp\!\!\!\perp h$ is a rational map with a period 2 cluster cycle. Then one of the angles associated with h has 2-angular rotation number $(q - p)/q$.*

Proof. This is similar to Lemma 2.4. The ray classes of the period 2 orbit $\{p_0, p_1\}$ of $f_{p/q}$ will become the cluster cycle and we note that the angle of any external ray landing on one of the p_i has 2-angular rotation number p/q. Let p_0 be the periodic point whose external rays all lie in $(2/3, 1/3)$ and label these angles in anticlockwise order (starting anywhere) by $\theta_1, \ldots, \theta_n$. We claim that $R^h_{-\theta_i}$ will land at the root point of a critical orbit Fatou component of h. If not, then the ray class would have to contain another periodic biaccessible point of $J(f_{p/q})$ and by Lemma 3.3, the only other biaccessible point is α_f, and by Lemma 1.7, $[\alpha_f] \neq [p_1] \neq [p_2] \neq [\alpha_f]$. By Lemma 1.1, one of the rays $R^h_{-\theta_i}$ will land at the root point of the critical value Fatou component of h, and so $-\theta_i$ will be one of the angles associated with h. Finally, as the angles in the orbit of θ_i have 2-angular rotation number p/q, the angles in the orbit of $-\theta_i$ will have 2-angular rotation number $(q - p)/q$. □

PROPOSITION 3.5. *All combinatorial data can be obtained.*

Proof. We will show that any combinatorial data $(\rho, \delta) = (p/q, 2k+1)$ can be obtained by a mating of the form $f_{p/q} \perp\!\!\!\perp h$. Clearly, if h is chosen appropriately, then the rational map formed by this mating will have a period 2 cluster cycle with combinatorial rotation number p/q, since matings preserve the combinatorial rotation number. Label the angles of the external rays landing at $p_0 \in J(f_{p/q})$ by $\theta_0, \dots, \theta_{q-1}$, where θ_0 is the angle of the external ray that approaches p_0 immediately anticlockwise from the critical point component of $f_{p/q}$. Then, to get $\delta = 2k+1$, let h be the map associated with the parameter ray of angle $-\theta_k$. Since $\theta_k \in (2/3, 1/3)$, it follows that $-\theta_k \in (2/3, 1/3)$ also, so the mating is not obstructed since the two polynomials do not belong to complex conjugate limbs of the Mandelbrot set. Also, since $-\theta_k$ is associated with h, the external ray of angle $-\theta_k$ lands at the base point of the critical value component of h. Thus, the rational map $F \cong f_{p/q} \perp\!\!\!\perp h$ exists and has combinatorial data (ρ, δ). □

A converse to Proposition 3.4 also exists if one places a suitable restriction on the angles that can be associated with the complementary map h.

Proposition 3.6. *Suppose that h has an associated angle θ which has 2-angular rotation number $(q-p)/q$ and that this angle lies in the arc $(2/3, 1/3) \subset S^1$. Then $F \cong f_{p/q} \perp\!\!\!\perp h$ and $F' \cong h \perp\!\!\!\perp f_{p/q}$ have a period 2 cluster cycle.*

Proof. Since $\theta \in (2/3, 1/3)$, the set $A_1 = \{2^{2k+1}\theta : k = 0, \dots, q-1\}$ is a subset of $(1/3, 2/3)$ and by definition has 1-angular rotation number p/q under the map $t \mapsto 4t$. Hence, the angles in A_1 correspond to the angles of the rays that land on the (unique) p/q-rabbit in $\mathcal{M}_4$ whose associated angles lie in $(1/3, 2/3)$. Hence, there are exactly q angles that satisfy the properties of the proposition and since, by Proposition 3.5, all q different critical displacements can be obtained, we are done. □

3.3. *Proof of Main Theorem* 2

We now prove the second main theorem. Again, we require a result from [**22**].

Theorem 3.7 [**22**, Theorem B]. *Suppose that two quadratic rational maps F and G have a period 2 cluster cycle with rotation number p/n and critical displacement δ. Then F and G are equivalent in the sense of Thurston.*

Proof of Main Theorem 2. By Proposition 3.5, all combinatorial data can be obtained and, by Theorem 3.7, maps with the same combinatorial data are Thurston equivalent, so all maps with period 2 cluster cycles are matings. By Proposition 3.2, one of the maps must be either the double rabbit $f_{p/q}$ or the unique period $2q$ polynomial $g_{p/q}$ that lies in the wake of $f_{p/q}$ in $\mathcal{M}$. Finally, by Proposition 3.4, the complementary map h must have an associated angle with 2-angular rotation number $(q-p)/q$. □

3.4. *Matings with the secondary map*

We now focus on the main difference between the matings in the fixed and period 2 cases, namely, that there exist non-trivial shared matings in the period 2 case. In the fixed case, one of the maps had to be a rabbit. However, it is not true in the period 2 case that one of the maps must be a double rabbit. In the case where neither of the maps is a double rabbit, we will show that one of the maps must instead be a certain polynomial that lies in the wake of the

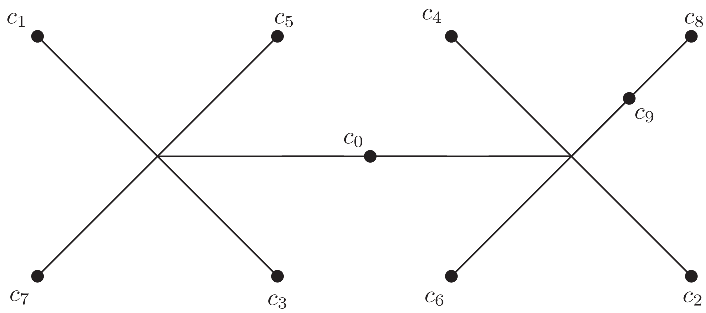

FIGURE 4. *Hubbard tree for a secondary map for $q = 5$, $\rho = 2/5$. Note, in particular, that c_{2q-1} lies on the branch from p_2 which has endpoint c_{2q-2}.*

double rabbit and has the same period as the double rabbit. We describe the general structure of the Hubbard trees of the secondary maps. This can be calculated from algorithms in [**2**], for example.

PROPOSITION 3.8. *The Hubbard tree of a secondary map can be described as follows. There are two period 2 points p_1 and p_2 with n arms. One of the global arms at p_1 contains the critical point c_0 and the other global arms at p_1 have endpoints $c_1, c_3, \ldots, c_{2q-3}$. The point p_2 has one global arm that contains the critical point c_0, and the endpoints of the other global arms are $c_2, c_4, \ldots, c_{2q-2}$. The point c_{2q-1} is on the arc (p_2, c_{2q-1}). Finally, there are no branch points on the arc (p_1, p_2), and c_0 lies on (p_1, p_2).*

As an example, Figure 4 shows the Hubbard tree for the secondary map that corresponds to the angles $(407/1023, 408/1023)$.

We now ask which rational maps can be obtained by matings with the secondary map. We will prove the following.

PROPOSITION 3.9. *Suppose $F \cong g_{p/q} \perp\!\!\!\perp h$ is a rational map. Then the critical displacement of F will be 1 or $2q - 1$.*

For the ease of notation, we shall drop the subscript on $g_{p/q}$. We will break down the proof into a sequence of lemmas, in which we will show that it is not possible for the rational map constructed by the mating $g \perp\!\!\!\perp h$ to have any other critical displacement apart from 1 or $2q - 1$. First, we make a comment on the angles of the rays that will be relevant to this discussion. The branch at the period 2 point p_1 in the Hubbard tree containing the critical value of g is separated from the other branches at the period 2 point by two rays, of angle θ and $\theta + 3$ (we suppress the denominator $2^{2q} - 1$). The rays of angles $\theta + 1$ and $\theta + 2$ land at the root point of the critical value component of g; see Figure 5.

We claim that the ray graph containing the point p_1 (as in the picture) must have (precisely) one of the following two properties. The branch of the graph of the ray-equivalence class containing the root point r of the critical value component U_1 of g (in other words, the landing point of the rays of angle $\theta + 1$ and $\theta + 2$) must contain either the ray of angle θ or the ray of angle $\theta + 3$. We will show that every other possibility is impossible.

LEMMA 3.10. *The branch of the ray-equivalence class containing r cannot contain the ray $R_g(\phi)$ which lands at p_1 and is such that $\phi \notin \{\theta, \theta + 3\}$.*

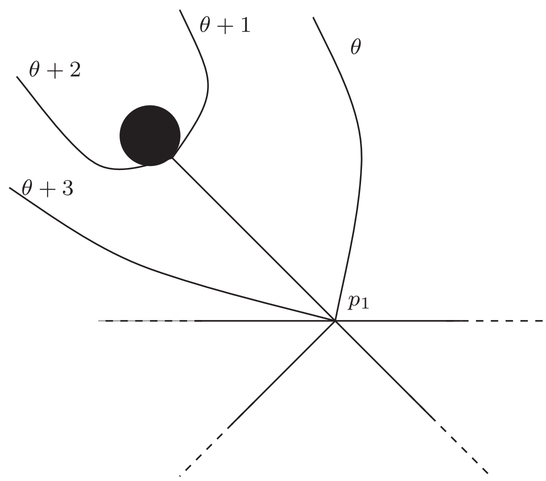

FIGURE 5. *The rays landing at the period 2 point and critical value component of the map g.*

Proof. It is clear that each branch of the graph must contain exactly one of the rays landing at the point p_1. If it contained more than one, then they would form a loop and thus would generate a Levy cycle. Suppose that the branch containing r contains a ray of angle $\phi \notin \{\theta, \theta+3\}$, and $R_g(\phi)$ lands at p_1. Let γ denote the (unique) path through external rays from r to p_1. Since the only possible biaccessible points in $J(g)$ on this path are r and p_1, we see that (using $[p, r]$ as the notation for the regulated arc from p to r)

$$\Gamma = \gamma \cup [p_1, r]$$

separates the sphere into two pieces. In particular, it separates the root point r' on the branch anticlockwise in the cyclic order from r from the ray of angle ϕ', which is the first angle anticlockwise round from ϕ in the cyclic ordering of rays at p. But since F has to maintain this cyclic ordering, the branch containing r' has to contain the ray of angle ϕ', and since rays cannot cross, this is a contradiction; see Figure 6. □

So we now know that the branch of the graph of the ray equivalence class containing r must contain the ray $R_g(\theta)$ or $R_g(\theta+3)$. We now study this branch in more detail.

LEMMA 3.11. (i) *The rays $R_h(-(\theta+1))$ and $R_h(-\theta)$ do not land at the same point on $J(h)$.*

(ii) *The rays $R_h(-(\theta+2))$ and $R_h(-(\theta+3))$ do not land at the same point on $J(h)$.*

Proof. Suppose that the ray $R_h(-(\theta+1))$ lands at the same point as $R_h(-\theta)$. This common landing point ζ cannot be the root point of a critical orbit component, since the size of the sector would mean that this would have to be a critical value component, and so the critical values would be in the same cluster, which is a contradiction of Lemma 1.1. However, the angular width of the sector bounded by ζ and the external rays $R_h(-(\theta+1))$ and $R_h(-\theta)$ is the smallest it can possibly be, and so it must contain the root point of the Fatou component that contains the critical point of h. But this is another contradiction since the root point of the critical value component needs to be in the ray-equivalence class of $f(p_1)$ and it is separated from this point. The proof of when $R_h(-(\theta+2))$ lands at the same point as $R_h(-(\theta+3))$ is analogous to the above. □

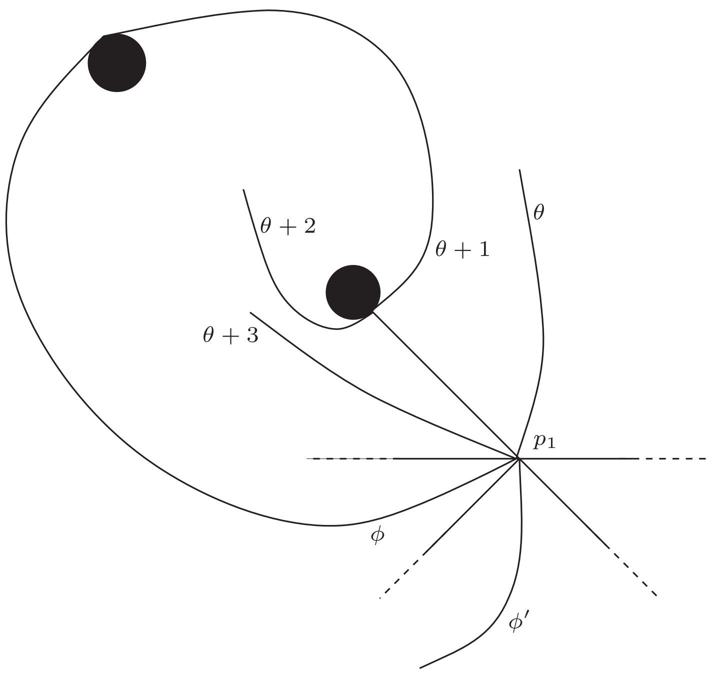

FIGURE 6. *The case where the branch containing r contains the ray of angle ϕ.*

Proof of Proposition 3.9. In light of Lemmas 3.10 and 3.11, the only remaining possibility is that the rays $R_h(-(\theta+2))$ and $R_h(-\theta)$ land at the same point or the rays $R_h(-(\theta+1))$ and $R_h(-(\theta+3))$ land at the same point. Since both situations are essentially the same, we shall discuss only the first one. The common landing point ξ of the two rays must be the root point of a critical orbit component. For if not, then we note that

$$R_g(\theta+2) \cup R_h(-(\theta+2)) \cup R_h(-\theta) \cup R_g(\theta)$$

separates the ray $R_h(-(\theta+1))$ from all other rays with the necessary denominator (Figure 7). Since the root point of a critical orbit component must have (at least) two rays landing on it, this means that this branch of the graph of the ray-equivalence class cannot contain a root point of a critical orbit component. But then none of the branches can, as each one maps homeomorphically onto its image and they are periodic. Hence, the rays $R_h(-(\theta+2))$ and $R_h(-\theta)$ land at the root point $\hat{r}$ of a critical orbit component. The angular width of the sector containing this component is 2, and so the angular width of its pre-image is 1. This means that the pre-image sector contains the critical value, and so in particular the pre-image of $\hat{r}$ is the root point of the critical value component. Hence, $\hat{r}$ is the root point of the component V_2 containing the second iterate of the critical point of h. Since V_2 is adjacent to U_1, it follows that their pre-images are adjacent, and so the critical point component of g will be adjacent to the critical value component of h. This means that the critical displacement of the resultant rational map will have to be ± 1, or equivalently, equal to 1 or $2n-1$. □

This proof shows us that, in order for a rational map to be a mating with the secondary map as one of the participants, one of the clusters has to have a critical point adjacent to the critical value of the other critical point.

COROLLARY 3.12. *Suppose $F \cong g_{p/q} \perp\!\!\!\perp h$ is a rational map with a period 2 cluster cycle. Then $F' \cong f_{p/q} \perp\!\!\!\perp h$ is also a rational map with a period 2 cluster cycle.*

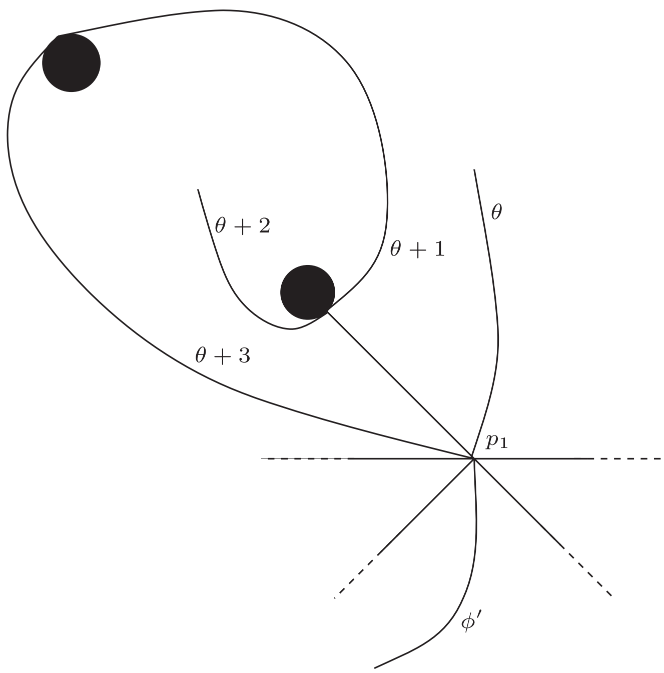

FIGURE 7. *How the ray class is formed near the critical value component of* $g_{p/q}$.

Proof. There are precisely two rays landing at the root points of the critical orbit Fatou components of h and the orbits of these rays are disjoint. Since F is a rational map, the angle θ associated with h must belong to $(2/3, 1/3)$. Furthermore, as F has a period 2 cluster cycle, one of these ray orbits is made up of angles of the form $-\theta_i$, where θ_i are the q angles of the rays that land on the period 2 orbit of $g_{p/q}$. But these are precisely the angles of the rays that land on the period 2 orbit of $f_{p/q}$. The orbit therefore has 2-angular rotation number p/q and so one of the angles associated with h satisfies the conditions of Proposition 3.6. □

The converse to Corollary 3.12 is not true. For example, if h is the quadratic polynomial associated with the angles $(13/63, 14/63)$, then $f_{1/3} \perp\!\!\!\perp h$ is equivalent to a rational map with combinatorial data $(1/3, 3)$. However, $g_{1/3} \perp\!\!\!\perp h$ is equivalent to a rational map that does not have a period 2 cluster cycle. We should now specify exactly how the two critical displacements $\delta = 1$ and $\delta = 2q - 1$ are obtained under the mating with the secondary map. Indeed, it is easy to see (for example, by considering 'drawing the rays shut' in Figure 7) that if we are in the case where $R_h(-\theta)$ and $R_h(-(\theta + 2))$ land at the same point, then $\delta = 1$. Similarly, if we are in the case where $R_h(-(\theta + 1))$ and $R_h(-(\theta + 3))$ land at the same point, then we will obtain $\delta = 2q - 1$ for the resultant rational map. Conversely, if $\delta = 1$ (respectively, $\delta = 2q - 1$), then the rays $R_h(-\theta)$ and $R_h(-(\theta + 2))$ (respectively, $R_h(-(\theta + 1))$ and $R_h(-(\theta + 3))$) land at the same point.

4. *Shared matings*

A little further work shows that if $g \perp\!\!\!\perp h$ has critical displacement 1 (respectively, $2n - 1$), then $f \perp\!\!\!\perp h$ has critical displacement $2n - 1$ (respectively, 1). The proof of this fact will allow us to discuss how the matings are shared in the period 2 cluster case.

Lemma 4.1. *The map $G \cong g_{p/q} \perp\!\!\!\perp h$ has critical displacement* 1 (*respectively*, $2q-1$) *if and only if the map $F \cong f_{p/q} \perp\!\!\!\perp h$ has critical displacement* $2q-1$ (*respectively*, 1).

Proof. By the observation at the end of the previous section, we see that if $G \cong g_{p/q} \perp\!\!\!\perp h$ has critical displacement 1, then the rays $R^h_{-\theta}$ and $R^h_{-(\theta+2)}$ land at the same point on $J(h)$, namely, the root point of U, the Fatou component that contains $h^{\circ 2}(0)$. So when we carry out the mating $f_{p/q} \perp\!\!\!\perp h$, we see that since $R^h_{-\theta}$ lands on the root point of U, the component U will lie first in the clockwise ordering of components from the critical value component corresponding to the critical point belonging to $f_{p/q}$. This means that the critical displacement will be $2q-1$, as required. The case for critical displacement $2q-1$ is similar.

Conversely, by the fact that the branches of the ray-equivalence classes $[p_1]$ and $[p_2]$ are all homeomorphic, it is only necessary to show that if $F \cong f_{p/q} \perp\!\!\!\perp h$ has a critical displacement of 1, then the ray class for $g_{p/q} \perp\!\!\!\perp h$ is as in Figure 7. By assumption, the ray $R_h(-(\theta+3))$ must land on the root point r of the critical orbit component of h containing $h^{\circ 2}(0)$, which we denote by V_2. We must show that the partner ray is $R_h(-(\theta+1))$. So let the partner ray be $R_h(\phi)$. Clearly, $\phi \neq -(\theta+2)$ since then V_2 would have to be the critical value component and $\phi \neq -\theta$ since this would create a loop in the ray-equivalence class, meaning that the mating would be obstructed. The union $R_h(-(\theta+3)) \cup R_h(\phi)$ separates $h^{\circ 2}(0)$ from the rest of the critical orbit of h (this is easy to see by, for example, considering the Hubbard tree of h). In particular, if $\phi > -(\theta+1)$, then we must have $-(\theta+3) < -(\theta+1) < \phi$, meaning that the ray $R_h(-(\theta+1))$ will not land on the root point of a critical orbit Fatou component of h. But this would contradict F having a period 2 cluster cycle, so we see that $\phi = -(\theta+1)$. In particular, this means that the ray class is as in Figure 7 and we are done. □

We now have all the information we need to start enumerating examples, except for a simple calculation as was promised after the definition of the period 2 critical displacement.

Lemma 4.2. *Suppose that the bicritical rational map F with labelled critical points has a period* 2 *cluster cycle with critical data* $(p/q, \delta)$. *Then under the alternative labelling, it has combinatorial data* $(p/q, 2p-\delta)$.

Proof. The critical displacement is by definition the combinatorial distance between c_1 and $F(c_2)$. By considering pre-images, this is the same as the distance between $F^{\circ(2q-1)}(c_1)$ and c_2. Since the combinatorial rotation number is p/q, we see that the distance between $F(c_1)$ and c_2 is $\delta - 2p$. Hence, the critical displacement when c_2 is taken to be the first critical point is $2p-\delta$. □

We see now that it is possible to get 2-fold, 3-fold and even 4-fold shared matings in the period 2 case. For example, we have the following equivalences of matings in the period 8, rotation number 1/4 case. The notation is as follows:

$f = f_{1/4}$ is the double rabbit corresponding to the angles 86/255 and 89/255;

$g = g_{1/4}$ is the secondary map corresponding to the angles 87/255 and 88/255;

h_1 is the map corresponding to the angles 83/255 and 84/255;

h_3 is the map corresponding to the angles 77/255 and 78/255;

h_5 is the map corresponding to the angles 53/255 and 54/255;

h_7 is the map corresponding to the angles 211/255 and 212/255.

With this notation, we have the following equivalences:

$$\begin{aligned}
(\delta = 1) &\quad f \perp\!\!\!\perp h_1 \cong g \perp\!\!\!\perp h_7 \cong h_1 \perp\!\!\!\perp f \cong h_7 \perp\!\!\!\perp g,\\
(\delta = 3) &\quad f \perp\!\!\!\perp h_3 \cong h_7 \perp\!\!\!\perp f \cong h_1 \perp\!\!\!\perp g,\\
(\delta = 5) &\quad f \perp\!\!\!\perp h_5 \cong h_5 \perp\!\!\!\perp f,\\
(\delta = 7) &\quad f \perp\!\!\!\perp h_7 \cong g \perp\!\!\!\perp h_1 \cong h_3 \perp\!\!\!\perp f.
\end{aligned}$$

Indeed, generally the smallest multiplicity of sharing is 2 and the greatest is 4. Using the notation above (so that h_δ is the polynomial such that $F \cong f_{p/q} \perp\!\!\!\perp h_\delta$ has combinatorial data (ρ, δ)), we always have the equivalence

$$f_{p/q} \perp\!\!\!\perp h_\delta \cong F \cong h_{2p-\delta} \perp\!\!\!\perp f_{p/q}. \tag{4.1}$$

Further, as we saw above, it is also sometimes possible to construct a rational map with a period 2 cluster cycle using a mating with the map $g_{p/q}$. In some cases (as with the case $\delta = 1$ above), this can give us a shared mating with multiplicity 4.

Proof of Main Theorem 3. All combinatorial data can be obtained by Proposition 3.5. By the observation of equation (4.1), each set of data can be obtained in at least two ways and as stated above, these data can be obtained with a mating with the map $g_{p/q}$ in at most two more ways. As shown in the calculation of the case where $\rho = 1/4$, all the possible multiplicities can be obtained. □

5. *Open questions*

We now ask how we may generalize the results of this paper. The first question is, even if we restrict ourselves to the quadratic case, what can we say about the matings that produce a map with a period p cluster cycle? Preliminary calculations suggest that there is far more complexity even when one steps up to the period 3 cluster cycle case. It is always possible to create such a rational map with the mating with an 'order p'-rabbit (the higher-order analogues to the rabbits and 'double rabbits' of this paper), but there are in general $2^{p-1} - 1$ 'secondary maps' in the period p case, each of which has a period p orbit with a well-defined combinatorial rotation number, and so is a viable candidate for participating in a mating that will create a rational map with a period p cluster cycle. Such considerations may lead to a better understanding of parameter space: some considerations of the fixed case (and using parabolic, not hyperbolic maps) can be found in Buff, Ecalle and Epstein [**3**]. Moreover, the question remains: how else can we create such rational maps and, assuming that there are other ways, how prevalent are the shared matings in the higher-period cases (compare with a question of Rees [**19**])? Of course, any approach towards answering this question needs to resolve two questions. Firstly, what matings can admit a map with a period p cluster cycle? Secondly, what is the analogue of the Thurston equivalence results as discussed in [**22**]? Of course, an attempt to understand the higher-degree cases would also be an interesting pursuit, since, as this paper and the sister paper show, even in the relatively benign period 2 case, the complications can cause Thurston equivalence to fail in higher degrees.

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