Math 562 Homework 5

Dr. Mark Comerford

Due Tuesday April 10, 2012

All problems worth 20 points, including bonus problems

1. *Singularities*

   For each of the following holomorphic functions which has an isolated singularity at $z = 0$, decide whether the singularity is removeable, a pole or essential. If it is a pole, find the singular part and if it is an essential singularity, determine the image of $D(0, \epsilon)$ where $\epsilon > 0$.

   (a) $f(z) = \frac{1}{1-e^z}$.

   (b) $f(z) = e^{1/z}$.

   (c) $f(z) = \cos(1/z)$.

   (d) $f(z) = \frac{\sin z}{z}$.

2. *Laurent Series*

   Let $f(z) = \frac{1}{(z-1)(z-2)}$. Find the Laurent series for f on each of the following:

   (a) $\mathbb{D}$;

   (b) $A(0, 1, 2)$;

   (c) $A(0, 2, \infty)$.

Please turn over!

3. *More Laurent Series*

Prove the following expansions:

(a)

$$e^z = e + e\sum_{n=1}^{\infty} \frac{(z-1)^n}{n!};$$

(b)

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \qquad z \in D(1,1);$$

(c)

$$\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n, \qquad z \in D(-1,1).$$

4. *Term-by-term Differentiation of Laurent Series*

Suppose $f(z)$ is holomorphic on an annulus $A = A(a,r,R)$ and has a Laurent series expansion $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ which converges on this annulus and uniformly on any proper subannulus as in the statement of Theorem 13.9. Show that we can differentiate term by term so that the Laurent series expansion for f' on A is $\sum_{n=-\infty}^{\infty} na_n(z-a)^{n-1}$.

Hint: This can be done by examining the proof of Theorem 13.10 and remembering how we split f into the sum of two functions f_1 and f_2. Now apply the corresponding result for Taylor series in the correct way to show that the formal derivative of f obtained by term-by-term differentiation also converges on A and uniformly on any proper subannulus. Now apply the result on term-by-term differentiation of Taylor series to f_1 and f_2 to obtain the desired conclusion.

5. *Approximation of $1/z$ by Polynomials*

Consider the function $f : A(0,r,R) \mapsto \mathbb{C}$, $f(z) = 1/z$ on an annulus with $0 \le r < R \le \infty$. Show that f is not the uniform limit of a sequence of polynomials on $A(0,r,R)$.