Math 562 Homework 4

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Due Tuesday March 27, 2012

All problems worth 20 points, including bonus problems

1. *Automorphisms of the Disc*

   Show that the biholomorphic maps (i.e. automorphisms) of the unit disc $\mathbb{D}$ are precisely the functions $\phi : \mathbb{D} \mapsto \mathbb{D}$ of the form

   $$\phi(z) = e^{i\theta}\frac{z - w}{1 - \overline{w}z}$$

   for some $\theta \in \mathbb{R}$ and some $w \in \mathbb{D}$.

   The proof consists of two main parts. The first part is to show that functions of this type are automorphisms of $\mathbb{D}$. First show that a map ϕ as above maps the unit circle $\mathbb{T}$ to itself. Now apply a theorem from the notes to conclude that $\phi(\mathbb{D}) \subset \mathbb{D}$. Then construct the inverse function ϕ^{-1} and use a similar argument to above to conclude that $\phi^{-1}(\mathbb{D}) \subset \mathbb{D}$. Next show that $\phi \circ \phi^{-1}$ and $\phi^{-1} \circ \phi$ are both the identity on $\mathbb{D}$. This shows that ϕ is a bijective map from $\mathbb{D}$ onto its range $\phi(\mathbb{D})$. Now deduce that in fact $\phi(\mathbb{D}) = \mathbb{D}$ and finally conclude that ϕ is a biholomorphic map of $\mathbb{D}$ onto itself.

   For the second part, let ϕ be any automorphism of $\mathbb{D}$. By composing ϕ with a suitable function, obtain another automorphism ψ of $\mathbb{D}$ for which $\psi(0) = 0$. Now apply the Schwarz lemma to ψ and ψ^{-1}.

2. *Uniform Convergence of Derivatives*

   Let Ω be a domain and let $\{f_n\}$ be a sequence of analytic functions on Ω which converges uniformly to f (which is then also analytic on Ω). Use the Schwarz lemma to show that the derivatives f_n' converge to f' and that this convergence is uniform on any compact subset of Ω.

   Hint: Consider the range of the differences $f_n - f$ and obtain from them a function $g : \mathbb{D} \mapsto \mathbb{D}$ which satisfies $g(0) = 0$.

Please turn over!

3. *The Schwarz-Pick Lemma*

   Let $f : \mathbb{D} \mapsto \mathbb{D}$ be analytic. Show that for every fixed $z_0 \in \mathbb{D}$ we have

$$|f'(z_0)| \leq \frac{1 - |f(z_0)|^2}{1 - |z_0|^2}.$$

   Hint: Pre- and postcompose with suitable automorphisms of $\mathbb{D}$ as found in Question 1 to obtain a function which fixes 0. Now apply the usual result about functions from $\mathbb{D}$ to itself which fix 0.

4. *Conformal Mappings Preserve Angles*

   Consider two differentiable curves γ and η which intersect at a point z_0 so that we can find t_0, t_1 such that

$$z_0 = \gamma(t_0) = \eta(t_1).$$

   We define the angle between the two curves to be the angle between the tangent vectors $\gamma'(t_0)$ and $\eta'(t_1)$.

   For two complex numbers z, w, we define the *scalar product* $< z, w >$ by

$$< z, w > := \mathrm{Re}\,(z\overline{w}).$$

   Let $\theta(z, w)$ be the angle between z and w (where we view z and w as vectors in the plane). Then

$$\cos\theta(z, w) = \frac{< z, w >}{|z||w|}.$$

   Since $\sin\theta = \cos(\theta - \pi/2)$, we also have

$$\sin\theta(z, w) = \frac{< z, -iw >}{|z||w|}.$$

   Now let f be an analytic function defined on a neighbourhood of z_0 with $f'(z_0) \neq 0$. Show that the angle between the curves $f \circ \gamma$ and $f \circ \eta$ is the same as that between γ and η.

   Hint: Use the chain rule to find the formula for $(f \circ \gamma)'(t)$.

   Does the result remain true if we allow the possibility that $f'(z_0) = 0$? Justify your answer!

Please turn over!

5. *An Estimate on the Number of Zeroes*

Consider a holomorphic map $f : \mathbb{D} \mapsto \mathbb{D}$ with $f(0) \neq 0$.

(a) Show that f has only finitely many zeroes z_j with $|z_j| \leq 1/3$ and call them $z_1, \ldots, z_n$. Show that (by defining it suitably at the points z_j) the function

$$g(z) := f(z) / \prod_{j=1}^{n} \left(\frac{z}{z_j} - 1 \right)$$

is holomorphic on $\mathbb{D}$.

Hint: For the sake of simplicity in this part, you may assume that all the points z_j are simple zeroes and that they are distinct. However, the same conclusion holds without these assumptions.

(b) Show that $|g(z)| \leq 2^{-n}$ for all $z \in \mathbb{D}$.

Hint: Show first that $|g(z)| \leq (3|z| - 1)^{-n}$ for all z with $|z| > 1/3$ and then apply the maximum principle.

(c) Show that

$$n \leq -\frac{\ln |f(0)|}{\ln 2}$$

where $\ln x$ is the usual natural logarithm from calculus.