Math 562 Homework 3

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Due Tuesday March 6, 2012

All problems worth 20 points, including bonus problems

1. *Taylor's Theorem with Analytic Remainder*

   Without using the Taylor Theorem from class show that for every analytic function $f : \Omega \mapsto \mathbb{C}$ on a domain $\Omega \subset \mathbb{C}$, every $a \in \Omega$ and every $n \in \mathbb{N}$, there is an analytic function $g_n : \Omega \mapsto \mathbb{C}$ such that

   $$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots\cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + g_n(z)(z-a)^n.$$

   Hints : Look at the proof of Taylor's Theorem, especially the algebraic trick and the formula for the Taylor coefficients a_n. Recall that for an analytic function g, the function h defined by

   $$h(z) = \begin{cases} \frac{g(z)-g(a)}{z-a} & : \quad z \neq a \\ g'(a) & : \quad z = a \end{cases}$$

   is analytic. The required construction and in particular the definitions of the functions g_n is by induction. Lastly, recall that Definition 4.10 as given in class states that a function g is analytic at z_0 if it has a local power series representation which converges on some disc of (positive) radius r about z_0. It then follows that g has derivatives of all orders at z_0 (why?).

Please turn over!

2. *Harder Path Integrals*

   (a) Evaluate for $n \geq 1$

$$\int_{C(0,1)} \frac{e^z}{z^n}\,\mathrm{d}z.$$

   Hint: Use the result in the notes for the path integral of a uniformly convergent sequence of functions.

   (b) Evaluate

$$\int_{C(0,2)} \frac{\mathrm{d}z}{z^2 - 4z + 3}.$$

   Hint: Try a partial fractions decomposition.

   (c) *(10 points extra credit) Evaluate

$$\int_{C(0,7)} \frac{\mathrm{d}z}{1 - e^z}.$$

   Hint: This one is tricky! You need to use the result in problem 1 (with $n = 1$) and apply another (modified) partial fractions decomposition.

3. *More Constant Entire Functions*

   Show that if f is an entire function for which we can find positive real constants A, B such that for every $z \in \mathbb{C}$,

$$|f(z)| \leq A + B\sqrt{|z|},$$

   then f must be a constant function.

   Hint: One way to do this is by looking at the proof of Liouville's Theorem. However, there are other ways!

Please turn over!

4. *Yet More Constant Entire Functions*

   Show that if f is a *non-constant* entire function, then the range of f must be dense in $\mathbb{C}$.

   Hint: Proceed by contradiction and suppose the result fails. Then we can find an open disc about some point which does not intersect the range of f. Now compose f with a suitable function and apply the usual theorem used for showing entire functions are constant!

5. *More Uniform Limits*

   Show that if f_n is a sequence of analytic functions defined on an open set $U \subset \mathbb{C}$ and f_n converges uniformly on U to a limit function f, then f must also be analytic on U.

   Hint: You will need one of the corollaries to Taylor's theorem, but which one?

6. Holomorphic k-th roots

   Let $\Omega \subset \mathbb{C}$ be a domain containing $a \in \mathbb{C}$ and $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function. A holomorphic function h is a k-th root of f near a if h is defined on a neighbourhood $U \subset \Omega$ of a and $f(z) = h(z)^k$ on U (here we take $k \geq 2$ to be an integer).

   (a) Assume $f(a) \neq 0$. Show that f has a k-th root near a.

   Hint: Use the logarithm.

   (b) Show that if f has a zero of order (multiplicity) $k \geq 2$ at a, then we can also find a k-th root of f near a.

   Hint: Use Taylor's Theorem and extract a factor $(z - a)^k$.

Please turn over!

7. * The Parseval Equation

Consider a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which is convergent on $D(0,R)$. Show that for all $0 < r < R$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \,\mathrm{d}\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Deduce from this the version of Cauchy's estimates

$$\frac{|f^{(n)}(0)|}{n!} = |a_n| \le \frac{M(r)}{r^n},$$

where $M(r) := \sup\{|f(z)| : |z| = r\}$.