Math 562 Homework 2

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Due Thursday February 16, 2012

All problems worth 20 points, including bonus problems

1. *The Cayley Mapping*

   Show that the function
   $$f(z) = i\frac{1+z}{1-z}$$
   maps the open unit disc $\mathbb{D}$ biholomorphically onto the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

   Hint: Find the inverse map! The inverse map to the function given above is called the *Cayley mapping*. To give a full correct answer, you will also need to show that f maps $\mathbb{D}$ into $\mathbb{H}$ while the inverse maps $\mathbb{H}$ into $\mathbb{D}$.

2. *The Exponential*

   (a) Show (for example by differentiating *repeatedly* with respect to z, comparing the power series and remembering that power series are unique) the functional equation for the exponential function:
   $$e^{z+w} = e^z e^w \qquad \text{for every} \quad z, w \in \mathbb{C}.$$
   (You may assume that every entire function has a power series.)

   (b) *(10 points extra) Show that the assumption that every entire function has a power series can be dispensed with if we use the fact that if a series $\sum a_n$ is absolutely convergent, then any rearrangement of the terms will also converge and have the same limit as the original.

Please turn over!

3. *The Logarithm*

On the slit plane $\mathbb{C}\backslash\mathbb{R}_0^- = \{z = re^{i\theta} : r > 0, -\pi < \theta < \pi\}$, we define the function $\operatorname{Log} : \mathbb{C}\backslash\mathbb{R}_0^- \to \mathbb{C}$ by $\operatorname{Log}(re^{i\theta}) := \log r + i\theta$. (This function is called the principal branch of the Logarithm and you have already seen it in class). Show that $\operatorname{Log} z$ is an inverse function to the exponential restricted to the strip $S = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi\}$, i.e. $\exp(\operatorname{Log} z) = z$ for every $z \in \mathbb{C} \setminus \mathbb{R}_0^-$. Conclude that $\operatorname{Log} z$ is holomorphic. Show that if z and w are in the right half plane (i.e. their real parts are positive), then

$$\operatorname{Log}(zw) = \operatorname{Log} z + \operatorname{Log} w.$$

4. *Bessel Functions*

For any integer $k \geq 0$, set

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left(\frac{z}{2}\right)^{2n+k}.$$

J_k is called the *Bessel Function of order k*.

(a) Find the radius of convergence of this series.
Hint: If a series $\sum_{n=0}^{\infty} a_n w^n$ has infinite radius of convergence, than so does the series $\sum_{n=0}^{\infty} a_n z^{2n}$. Using this trick, you can still use the ratio test to calculate the radius of convergence!

(b) Show that J_k satisfies the differential equation

$$z^2 J_k''(z) + zJ_k'(z) + (z^2 - k^2)J_k(z) = 0.$$

This differential equation is known as *Bessel's equation*.

Please turn over!

5. *Some Path Integrals*

(a) Find

$$\int_\gamma (z + \overline{z})\,\mathrm{d}z$$

where γ is given by $\gamma(t) = e^{-\pi i t}$, $0 \le t \le 1$.

(b) Find

$$\int_\gamma \overline{z}\,\mathrm{d}z$$

where γ is given by $\gamma(t) = 1 + it$, $0 \le t \le 1$.

(c) Find

$$\int_\gamma z^5\,\mathrm{d}z$$

where γ is given by $\gamma(t) = 1 + it + t^2$, $0 \le t \le 1$.

6. *Some more Path Integrals*

Find

$$\int_\gamma z e^{z^2}\,\mathrm{d}z$$

(a) Where γ is the section of the parabola $y = x^2$ which runs from 0 to $1 + i$.

(b) Where γ is the straight line segment which runs from i to $2 - i$.

Please turn over!

7. **An Example of an Analytic Function.*

(a) Show (by differentiating twice) that for all $z \in \mathbb{D}$, we have

$$\exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n}\right) = 1 - z$$

and deduce that

$$\mathrm{Log}\,(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$$

where $\mathrm{Log}\, z$ is the function of problem 3.

(b) Show that for all $z \in \mathbb{D}$, we have

$$\begin{aligned} \mathrm{Log}\,(1+z+z^2) &= z + \frac{1}{2}z^2 - 2\frac{1}{3}z^3 + \frac{1}{4}z^4 + \frac{1}{5}z^5 - 2\frac{1}{6}z^6 \pm \cdots \\ &= \sum_{n=1}^{\infty} a_n z^n, \quad \text{where} \quad a_n = \begin{cases} -2 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \equiv 1, 2 \pmod 3 \end{cases} \end{aligned}$$

Hint: There is a complex number ω with

$$1 + z + z^2 = (1 - \omega z)(1 - \overline{\omega} z).$$