Math 562 Homework 1

Dr. Mark Comerford

Due Tuesday February 7, 2012

All problems are worth 20 points, including bonus problems which are extra credit.

1. *Complex Square Roots*

For a complex number $z = a+ib$, find a formula for the two possible complex square roots of z, i.e. the complex numbers w for which $w^2 = z$.

2. *Coordinate Grid Under* z^2

Consider the coordinate grid $G = \{z = x + iy : x \in \mathbb{Z} \vee y \in \mathbb{Z}\}$.

1. Show that the image of G under z^2 gives two families of parabolas.
2. Show that the inverse image of G under z^2 gives two families of hyperbolas.

There will be five points of extra credit for particularly nice pictures! The use of Mathematica and Maple is encouraged!

3. *The Cauchy-Riemann Equations in Polar Coordinates*

Let $w = f(z) = u(z) + iv(z)$ be a holomorphic complex function.

1. Show that if we regard u and v as functions of r and θ, the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

2. Now write $w = f(z) = R\cos(\Theta) + iR\sin(\Theta)$.

   (a) Show that if we regard R and Θ as functions of x and y, the Cauchy-Riemann equations become

$$\frac{1}{R}\frac{\partial R}{\partial x} = \frac{\partial \Theta}{\partial y}, \qquad \frac{1}{R}\frac{\partial R}{\partial y} = -\frac{\partial \Theta}{\partial x}.$$

(b) Show that if we regard R and Θ as functions of r and θ, the Cauchy-Riemann equations become

$$\frac{1}{R}\frac{\partial R}{\partial r} = \frac{1}{r}\frac{\partial \Theta}{\partial \theta}, \qquad \frac{1}{R}\frac{\partial R}{\partial \theta} = -r\frac{\partial \Theta}{\partial r}.$$

4. Show that if the absolute value $|f|$ of a holomorphic function f is constant, then f itself must be constant.

5. *Power Series and the Ratio Test*

Show that for a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, if the limit $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists and has value r, then the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

is $1/r$ (with the radius of convergence being infinite in the case $r = 0$).

Why can't we use this simpler criterion as a definition for the radius of convergence? Explain your answer!

6. Find the radii of convergence of the following power series:

1.
$$\sum_{n=0}^{\infty} 3^n z^n;$$

2.
$$\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n;$$

3.
$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} z^n.$$

7. *One for all the Difference Equations People*

Consider the sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ defined by

$$a_1 = 1, a_2 = 2, \qquad a_n = a_{n-1} + a_{n-2}, \quad n \geq 3.$$

Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} a_n z^n.$$

Hint: Try looking at the ratio between successive terms of the sequence. Does this have a limit?

8.* Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

1. Show that the series diverges at $z = 1$.
2. Show that the series converges at all other values of z on the unit circle.

   Hint: Consider the partial sums of this series from a geometric point of view.