§ 2.7 Independent Events

Informally, two events are independent if the occurrence or non-occurrence of either one does not influence the probability of the other. Formally we have

Defn 2.2 Two events A and B are independent iff

$$P(A \cap B) = P(A) \cdot P(B).$$

Note that by Theorem 2.9, if $P(A) \neq 0$, then

$$P(A \cap B) = P(A) P(B|A)$$

$$P(A) \cdot P(B) = P(A) P(B|A)$$

So $P(B) = P(B|A)$ (ok as $P(A) \neq 0$).

Similarly, if $P(B) \neq 0$, then

$$P(A) = P(A|B).$$

If two events A & B are not independent, we say they are dependent.

<u>Ex</u>. Let A, B be two events and suppose $P(A)=0$. Then A and B are indendent.

To see this note first that $A \cap B \subset A$, so
$0 \leq P(A \cap B) \leq P(A) = 0$, by Postulate 1 & Thm 2.5
and so $P(A \cap B) = 0$.

Thus $P(A \cap B) = 0 = 0 \cdot P(B) = P(A) \cdot P(B)$.

Similarly, if $P(B) = 0$, then A and B are automatically independent.

Ex. Coin is tossed 3 times and each of the 8 possible outcomes has prob. $\frac{1}{8}$.

Let

$$A = \left\{ \begin{array}{l} \text{head occurs on each of the} \\ \text{first two tosses} \end{array} \right\}$$

$$B = \{ \text{tail occurs on third toss} \}$$

$$C = \{ \text{exactly 2 tails occur in the 3 tosses} \}.$$

Show

a) A & B are indep.

b) B & C are dependent.

Soln

$$A = \{HHH, HHT\}$$

$$B = \{HHT, HTT, THT, TTT\}.$$

$$C = \{HTT, THT, TTH\}$$

$$A \cap B = \{HHT\}$$

$$B \cap C = \{HTT, THT\}.$$

$P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{3}{8}$,

$P(A \cap B) = \frac{1}{8}$, $P(B \cap C) = \frac{1}{4}$

a) $P(A \cap B) = \frac{1}{8}$

$P(A)P(B) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$,

So A, B are indep.

b) $P(B \cap C) = \frac{1}{4}$

$P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16} \neq \frac{1}{4}$,

So B, C are dep.

<u>Thm 2.11</u> If A, B are indep., then A, B^c are indep.

Pf. Since $A = (A \cap B) \cup (A \cap B^c)$ and this union is disjoint

$$P(A) = P((A \cap B) \cup (A \cap B^c))$$
$$= P(A \cap B) + P(A \cap B^c)$$

Thus $P(A \cap B^c) = P(A) - P(A \cap B)$

$= P(A) - P(A) \cdot P(B)$ by ind. of A, B

$= P(A)(1 - P(B))$

$= P(A) P(B^c)$ as $P(B^c) = 1 - P(B)$.

$\blacksquare$

Independence for more than two events.

<u>Defn</u> 2.3 Events $A_1, A_2, \ldots, A_k$ are <u>independent</u> if the probability of the intersection of any $2, 3, \ldots, k$ events is the product of their respective probabilities

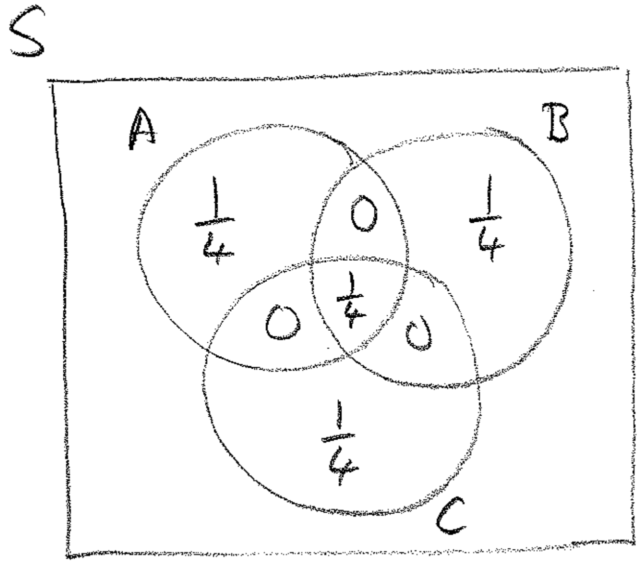
i.e. for each $2 \le i \le k$, and each

$$1 \le n_1 < n_2 < n_3 < \cdots < n_i \le k$$

$$P\left(\bigcap_{j=1}^{i} A_{n_j}\right) = \prod_{j=1}^{i} P(A_{n_j})$$

As the next example shows, 3 or more events can be <u>pairwise</u> <u>independent</u> without actually being independent.

Ex. Consider the following Venn diagram with probabilities assigned to the various regions



From the diagram

$$P(A) = P(B) = P(C) = \frac{1}{2}$$

$$P(A \cap B) = P(B \cap C) = P(A \cap C) = \frac{1}{4}$$

$$P(A \cap B \cap C) = \frac{1}{4}$$

Thus

$$P(A) \cdot P(B) = \frac{1}{4} = P(A \cap B)$$

$$P(A) \cdot P(C) = \frac{1}{4} = P(A \cap C)$$

$$P(B) \cdot P(C) = \frac{1}{4} = P(B \cap C)$$

but

$$P(A) \cdot P(B) \cdot P(C) = \frac{1}{8} \neq \frac{1}{4} = P(A \cap B \cap C).$$

Can also happen that $P(A) \cdot P(B) \cdot P(C) = P(A \cap B \cap C)$ but A, B, C not indep (see hw).

Ex. Find the probs. of getting.

a) 3 heads in 3 random tosses of a balanced coin

b) four sixes and then another number (ie not a six) in 5 random rolls of a balanced die.

Soln: Multiply probabilities to get.

a)

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

b)

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{7,776}$$