2.6

# Conditional Probability

Conditional probability is one of the most important ideas in this course. Basically, it measures how one event may influence another. To put it slightly differently, our choice of sample space will affect the probability of an event.

Ex. During the South Carolina primary, Mike Huckabee polled 12% among South Carolina republicans generally and 40% among those republicans who identified themselves as evangelicals.

Thus the probability that a random South Carolina republican would vote for Huckabee is different for the sample spaces

$S_1$ = {all South Carolina republicans},

$S_2$ = {evangelical South Carolina republicans}.

Note that here $S_2 \subset S_1$. This is a common feature where one sample space is a restriction of (ie smaller than) another.

Ex. Push Polling.

Suppose you are a registered democrat. A pollster calls you to ask if you would support Obama for the democratic nomination.

Now suppose another pollster calls you. They start by asking you if you were aware of how Senator Hilary Clinton and Former President Bill Clinton have misquoted remarks of Senator Obama in order to claim that he secretly endorses many of the republicans' policies.

Would you be more likely to tell the second pollster you'll support Obama?

What the push pollster is trying to do is change the sample space as much as possible in order to obtain a more favourable result for Obama.

We use the notation $P(A|S)$ for the conditional probability of $A$ given $S$, so that the sample space $S$ is made explicit.

Ex. A survey of 50 car dealerships in a city gave the following results

| | Good service under warranty | Poor service under warranty |
|---|---|---|
| In business $\geq 10$ years | 16 | 4 |
| In business $< 10$ years | 10 | 20 |

What are the chances a person gets good service under warranty

a) from any randomly chosen dealer?

b) from a randomly chosen dealer who has been in business $\geq 10$ years?

Sol$^n$. 'Randomly' here means equally likely and so we can use Thm 2.2 $\left( \frac{\#\text{ favourable}}{\#\text{ possible}} \right)$

Let $G$ denote the event 'good service' and $T$ denote the event 'has been in business $\geq 10$ years'.

a) Want $P(G)$. By Thm 2.2.

$$P(G) = \frac{n(G)}{n(S)} = \frac{16+10}{50} = 0.52$$

b) Here we have the restricted sample space given by $T$ which contains $16+4=20$ dealerships and we want $P(G|T)$.
Again by Thm 2.2

$$P(G|T) = \frac{16}{20} = 0.8.$$

Note that 16 here is the number of elements in $T \cap G$ and so

$$P(G|T) = \frac{n(T \cap G)}{n(T)}.$$

If we divide above & below by $n(S)$ we get

$$P(G|T) = \frac{\frac{n(T \cap G)}{n(S)}}{\frac{n(T)}{n(S)}} = \frac{P(T \cap G)}{P(T)}$$

by Thm 2.2.

We have expressed $P(G|T)$ in terms of two other probabilities defined for the whole sample space $S$. This motivates the following defn.

Defn 2.1 If $A$ & $B$ are any two events in a sample space $S$ and $P(A) \neq 0$, the conditional probability of $B$ given $A$ is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Ex. In the previous example, what is the prob. that one of the dealers who has been in service < 10 years provides good service?

Soln. Want $P(G \mid T^c)$

Now $P(T^c \cap G) = \dfrac{10}{50} = .2$

and $P(T^c) = \dfrac{10+20}{50} = .6$ and so

$$P(G \mid T^c) = \frac{P(T^c \cap G)}{P(T^c)} = \frac{.2}{.6} = \frac{1}{3}.$$

Ex. Loaded die of an earlier example where the odd nos. had prob $\frac{2}{9}$ each and the even nos. had prob. $\frac{1}{9}$ each.

Roll the die once.

a) What is the prob. that the number rolled is a perfect square?

b) What is the prob. that the number rolled is a perfect square given that it is greater than 3?

Soln. Let $A = \{\text{no. rolled is} > 3\}$

$B = \{\text{no. rolled is a perfect square}\}$

a) Want $P(B)$. $B = \{1, 4\}$ and so

$$P(B) = P(1) + P(4) = \frac{2}{9} + \frac{1}{9} = \frac{1}{3}$$

b) Want $P(B \mid A)$

$$A = \{4, 5, 6\} \text{ and } P(A) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$$

while $B \cap A = \{4\}$ and $P(B \cap A) = \frac{1}{9}$.

Then

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{9}}{\frac{4}{9}} = \frac{1}{4}.$$

Ex. For a factory making aircraft parts the prob that an order is ready for shipment on time is 0.8 and the prob. that it is ready for shipment on time and is delivered on time is .72. What is the prob. the order is delivered on time given that it was ready for shipment on time?

Soln. Let $R = \{\text{ready for shipment on time}\}$

$D = \{\text{delivered on time}\}$.

Have $P(R) = 0.8$, $P(R \cap D) = .72$

Want $$P(D|R) = \frac{P(R \cap D)}{P(R)} = \frac{.72}{.8} = .9.$$

We defined conditional probability by

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad (P(A) \neq 0)$$

If we multiply both sides by $P(A)$, we get

<u>Theorem 2.9</u> If $A$ and $B$ are any two events in a sample space $S$ and $P(A) \neq 0$ then

$$P(A \cap B) = P(A) \cdot P(B|A).$$

Note that we can swap the roles of $A$, $B$ & use the fact that $B \cap A = A \cap B$ to get

$$P(A \cap B) = P(B) \cdot P(A|B).$$

$(P(B) \neq 0)$.

Ex. Pick two television tubes from a shipment of 240 of which 15 are defective. What is the chance that both tubes are defective?

Soln. We assume we are sampling without replacement.

Prob first tube is defective is $\frac{15}{240}$

and prob second tube is defective is then $\frac{14}{239}$. Thus prob both tubes are defective is

$$\frac{15}{240} \times \frac{14}{239} = \frac{7}{1912}$$

Ex. What are the chances of randomly drawing two aces in succession from a pack of 52 playing cards if we sample

a) without replacement?

b) with replacement?

Soln. a) Without replacement, the prob is

$$\frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

b) With replacement, the prob. is

$$\frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$$

Q. Explain why the second prob. is larger than the first.

We can generalize Thm 2.9 to more than 2 events.

<u>Thm 2.10</u> If A, B, C are events in a sample space S, then

$$P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot P(C \mid A \cap B)$$

P.F. Write $A \cap B \cap C$ as $(A \cap B) \cap C$. By Thm 2.9 (twice)

$$\begin{aligned} P(A \cap B \cap C) &= P((A \cap B) \cap C) \\ &= P(A \cap B) \cdot P(C \mid A \cap B) \\ &= P(A) \cdot P(B \mid A) \cdot P(C \mid A \cap B). \quad \boxed{} \end{aligned}$$

<u>Ex.</u> A box has 20 fuses with 5 being defective. If we take 3 fuses without replacement, what is the prob. that all 3 are defective?

<u>Soln.</u> Let $A = \{$1st fuse defective$\}$
$B = \{$2nd fuse defective$\}$
$C = \{$3rd fuse defective$\}$.

Want $P(A \cap B \cap C)$.

Here $P(A) = \frac{5}{20}$, $P(B \mid A) = \frac{4}{19}$, $P(C \mid A \cap B) = \frac{3}{18}$.

Then

$$P(A \cap B \cap C) = \frac{5}{20} \cdot \frac{4}{19} \cdot \frac{3}{18}$$

$$= \frac{1}{114}.$$