2.5 Some Rules of Probability

Thm 2.3 If A, A^c are complementary events in a sample space, then $P(A^c) = 1 - P(A)$.

Pf. $S = A \cup A^c$ and this union is disjoint.

$$P(S) = P(A \cup A^c) = P(A) + P(A^c) \qquad \text{by postulate 3}$$

$$1 = P(A) + P(A^c) \qquad \text{by postulate 2}$$

$$1 - P(A) = P(A^c)$$

$$P(A^c) = 1 - P(A) \qquad \square$$

Cor 2.4 $P(\emptyset) = 0$ for any sample space S.

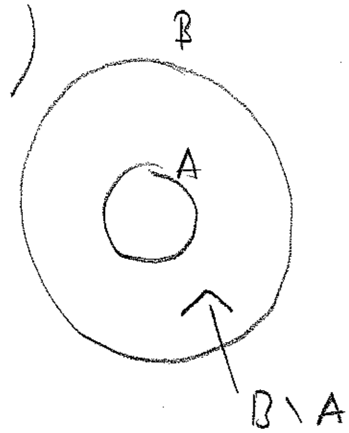
Pf. Let $A = S$ and apply Thm 2.3. $\square$

Nb. The book uses A' to denote the complement of an event, but A^c is the more standard notation.

<u>Thm 2.5</u> If A, B are events in a sample space and $A \subseteq B$, then $P(A) \leq P(B)$.

Pf. Since $A \subseteq B$, we can write B as

$$B = A \cup (B \setminus A) \quad (= A \cup (B \cap A^c))$$



and since this union is disjoint, by Postulate 3

$$P(B) = P(A) + \underbrace{P(B \setminus A)}_{\geq 0}$$

$$\geq P(A) \quad \text{by Postulate 1} \qquad \square$$

<u>Cor 2.6</u> $0 \leq P(A) \leq 1$ for any event A.

Pf. $\emptyset \subseteq A \subseteq S$, so by Thm 2.5

$$P(\emptyset) \leq P(A) \leq P(S)$$

ie $0 \leq P(A) \leq 1$ by Postulate 2 & Cor 2.4

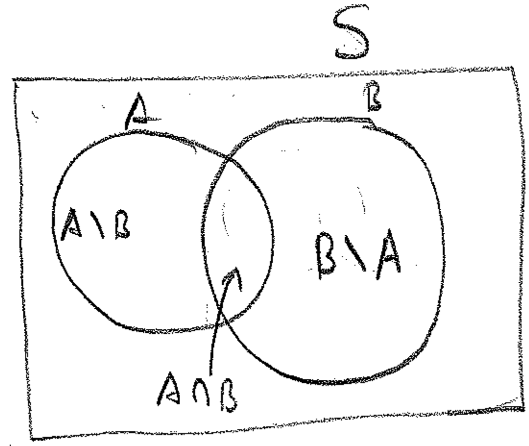
$\square$

Thm 2.7 (Addition Rule).

If A, B are events in a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Pf. $A \cup B$ can be expressed as a pairwise disjoint union



$$A \cup B = A \setminus B \cup A \cap B \cup B \setminus A$$

and so by Postulate 3

$$P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$$

$$= P(A \setminus B) + P(A \cap B) + P(B \setminus A) + P(A \cap B) - P(A \cap B)$$

trick!

$$= P(A) + P(B) - P(A \cap B)$$

by Postulate 3 again as

$$A = A \setminus B \cup A \cap B$$

$$B = B \setminus A \cup A \cap B$$

are disjoint unions $\square$.

Ex. In Chicago, the probabilities are 0.86, 0.35, 0.29 that a randomly chosen family owns a colour TV, a HDTV, or both (respectively). What is the prob. that a family owns either or both kind of TV set?

Soln Let $A = \{\text{family owns a colour TV}\}$

$B = \{\text{family owns a HDTV.}\}$

Then $A \cap B = \{\text{family owns both types of TV}\}$.

and we want $P(A \cup B)$.

By Thm 2.7,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= .86 + .35 - .29$$

$$= .92.$$

Ex. On I-95, Exit 2A,

prob. that a truck stopped by the police has faulty brakes is $.23$ and that it has badly worn tyres is $.24$ while the prob that the truck has faulty brakes and/or worn tyres is $.38$. What is the prob. that a truck has both faulty brakes and worn tyres?

Soln. Let $F = \{\text{truck has faulty brakes}\}$.

$T = \{\text{truck has worn tyres}\}$.

We know $P(F) = .23$, $P(T) = .24$,

$$P(F \cup T) = .38.$$

Want $P(F \cap T)$.

By Thm 2.7

$$P(F \cup T) = P(F) + P(T) - P(F \cap T)$$

$$.38 = .23 + .24 - P(F \cap T)$$

So $P(F \cap T) = .23 + .24 - .38 = .09$ (9%).

We can generalize Thm 2.7 to three or more events.

Thm 2.8 If A, B, C are events in a sample space S, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

First we prove a lemma (which is an exercise in the book)

Lemma $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- distributive property.

Pf. A common way of showing two sets (events) E, F are the same set is to first show $E \subseteq F$ and then $F \subseteq E$ which then implies that $E = F$.

So suppose $x \in A \cap (B \cup C)$.

Then $x \in A$ and $x \in B \cup C$.

Since $x \in B \cup C \Rightarrow x \in B$ <u>or</u> $x \in C$.
(or both).

Then if $x \in B$, $x \in A$ also from above

so $x \in A \cap B \subset (A \cap B) \cup (A \cap C)$.

Similarly if $x \in C$, $x \in A$ also,

so $x \in A \cap C \subset (A \cap B) \cup (A \cap C)$.

In either case $x \in (A \cap B) \cup (A \cap C)$, so since x was arbitrarily chosen, we must have

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

Now conversely suppose that $x \in (A \cap B) \cup (A \cap C)$.

Then either $x \in A \cap B$ <u>or</u> $x \in A \cap C$ (or both).

If $x \in A \cap B$, then $x \in A$ <u>and</u> $x \in B$,

so $x \in B \cup C$ also. Hence $x \in A \cap (B \cup C)$.

Similarly, if $x \in A \cap C$, then $x \in A$ and $x \in B$, so $x \in B \cup C$ also. Hence $x \in A \cap (B \cup C)$.

In either case $x \in A \cap (B \cup C)$, so since x was arb. chosen, we must have

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

Thus $\quad A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and

$$A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$$

and so

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

as required. $\square$

Now back to business...

<u>Pf of Thm 2.8.</u> Write $A \cup B \cup C$ as $A \cup (B \cup C)$ & apply Thm 2.7 to get

$$P(A \cup B \cup C) = P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap (B \cup C)) \quad \text{by Thm 2.7 on } B, C.$$

$$= P(A) + P(B) + P(C) - P(B \cap C)$$

$$- \left(P((A \cap B) \cup (A \cap C)) \right) \quad \text{by the Lemma}$$

$$= P(A) + P(B) + P(C) - P(B \cap C)$$

$$- \left(P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) \right).$$

$$= P(A) + P(B) + P(C) - P(B \cap C)$$

$$- P(A \cap B) - P(A \cap C) + P((A \cap B) \cap (A \cap C))$$

$$= P(A) + P(B) + P(C)$$

$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$

$$+ P(A \cap B \cap C) \quad \text{as clearly}$$

$$(A \cap B) \cap (A \cap C)$$

$$= A \cap B \cap C.$$

So $P(A \cap B \cap C) = P(A) + P(B) + P(C)$

$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$

$$+ P(A \cap B \cap C).$$

Ex. Girls at Roedean play hockey, netball, and lacrosse. We are given the following information that for a randomly chosen girl,

P(plays hockey) = .6

P(plays netball) = .5

P(plays lacrosse) = .45

P(plays hockey and netball) = .25

P(plays hockey and lacrosse) = .2

P(plays netball and lacrosse) = .25

P(plays at least one sport) = .9.

What is the prob a randomly chosen girl plays

a) all three sports

b) hockey only.

c) netball and hockey only.

Soln. Let H, N, L stand for the events that a girl plays hockey, netball, lacrosse resp.

We are given

$$P(H) = .6, \quad P(N) = .5, \quad P(L) = .45$$

$$P(H \cap N) = .25, \quad P(H \cap L) = .2, \quad P(N \cap L) = .25$$

$$P(H \cup N \cup L) = .9.$$

a). By Thm 2.8

$$\begin{aligned} P(H \cup N \cup L) = {} & P(H) + P(N) + P(L) \\ & - P(H \cap N) - P(H \cap L) - P(N \cap L) \\ & + P(H \cap N \cap L). \end{aligned}$$

$$\begin{aligned} .9 = {} & .6 + .5 + .45 - .25 - .2 - .35 \\ & + P(H \cap N \cap L) \end{aligned}$$

So

$$P(H \cap N \cap L) = .9 - .6 - .5 - .45 + .25 + .2 + .25$$

$$= .05.$$

b) Hockey only is represented by the event $H \setminus (N \cup L)$.

Now $H = (H \cap (N \cup L)) \cup (H \setminus (N \cup L))$ and this union is disjoint, so by Postulate 3,

$$P(H) = P(H \cap (N \cup L)) + P(H \setminus (N \cup L))$$

$$= P((H \cap N) \cup (H \cap L)) + P(H \setminus (N \cup L))$$ by Lemma

$$= P(H \cap N) + P(H \cap L) - P((H \cap N) \cap (H \cap L)) + P(H \setminus (N \cup L))$$ by Thm 2.7

$$= P(H \cap N) + P(H \cap L) - P(H \cap N \cap L) + P(H \setminus (N \cup L))$$ since $(H \cap N) \cap (H \cap L) = H \cap N \cap L$.

We have now reduced to the stage where we know all the probs. except that of the event we want. So

$$\cdot 6 = \cdot 25 + \cdot 2 - \cdot 05 + P(H \setminus (N \cup L))$$

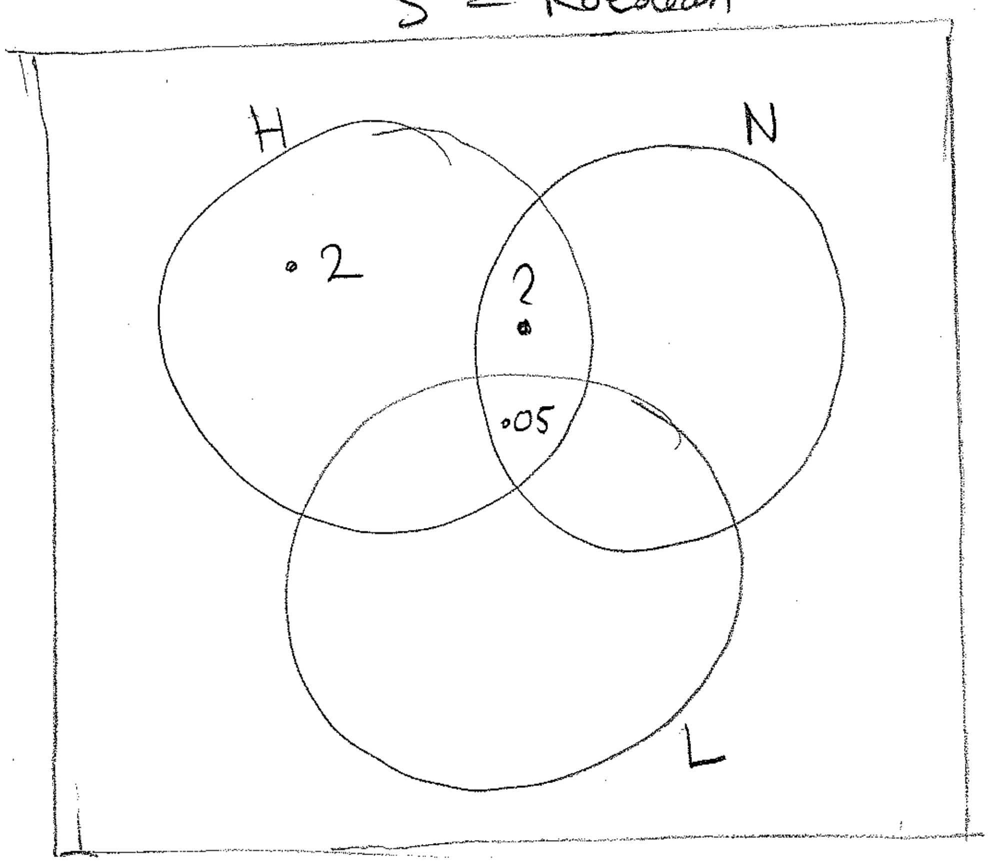
So $P(H \setminus (N \cup L)) = \cdot 6 - \cdot 25 - \cdot 2 + \cdot 05$

$$= \cdot 2.$$

c) Hockey & Netball only is represented by the event

$(H \cap N) \setminus L$

S = Roedean



Now

$$H \cap N = ((H \cap N) \cap L) \cup ((H \cap N) \setminus L)$$

$$= (H \cap N \cap L) \cup ((H \cap N) \setminus L)$$

and this union is disjoint, so by Postulate 3.

$$P(H \cap N) = P(H \cap N \cap L) + P((H \cap N) \setminus L)$$

$$.25 = .05 + P((H \cap N) \setminus L)$$

$$P((H \cap N) \setminus L) = .25 - .05 = .2$$

Ex. Fill in the rest of the Venn diagram.

We can obviously intersect any number of events. FYI the formula in the general case is

<u>Thm</u> If $A_1, A_2, \ldots, A_n$ are n events in a sample space S, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n}\left((-1)^{i-1}\sum_{1\le j_{i,1}<j_{i,2}<\cdots<j_{i,i}\le n} P\left(\bigcap_{k=1}^{i} A_{j_{i,k}}\right)\right).$$

I DO NOT expect you to memorise this!