Chapter 4 Mathematical Expectation

§4.1, 4.2 Introduction and the Expectation of a Random Variable

Basically the expectation of a random variable is the average or expected value of the r.v.

e.g. I play a game in which I roll a (fair) die.

If I roll a 1, I win \$2

If I roll a 2 or a 3, I win \$1

If I roll a 4, 5, or 6, I lose \$1.

On average (i.e. if I play the game many times), I would expect to win.

$$2 \times \frac{1}{6} + 1 \times \frac{1}{3} - 1 \times \frac{1}{2} = \$\frac{1}{6}$$

$$\approx 17¢.$$

What we did was multiply the value of each outcome by the prob of that outcome and then sum over all possible outcomes.

<u>Defn</u>. If X is a discrete r.v. with pdf $f(x)$, the expected value of X is

$$E(X) = \sum_{x} x \cdot f(x).$$

If X is a cts r.v. with density $f(x)$, the <u>expected value</u> of X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

N.b. we assume here, of course that the sum and integral are defined.

It is in fact possible that a r.v. can have infinite expectation!

<u>Remark</u> For the physicists among you, note the similarity with centre of mass and also the position operator in quantum mechanics!

<u>Ex.</u> A lot of 12 tvs includes 2 with white cords. If a shipment includes 3 tvs chosen at random, how many can one expect to have white cords.

<u>Soln</u> Let X = # tvs shipped with white cords.

If we choose x tvs with white cords where $x = 0, 1, 2$ then there are $\binom{2}{x}$ ways of choosing the x tvs with white cords and $\binom{10}{3-x}$ ways of choosing the remaining $3-x$ which don't have white cards.

Since there are $\binom{12}{3}$ ways of choosing the 3 tvs altogether, if we let $f(x)$ be the pdf of X, then by Thm 2.2

$$f(x) = \frac{\binom{2}{x}\binom{10}{3-x}}{\binom{12}{3}}, \quad x = 0, 1, 2$$

and $f(x) = 0$ otherwise.

Can make a table

x	0	1	2
$f(x)$	$\frac{6}{11}$	$\frac{9}{22}$	$\frac{1}{22}$.

Then

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2)$$

$$= 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2)$$

$$= 0 \cdot \frac{6}{11} + 1 \cdot \frac{9}{22} + 2 \cdot \frac{1}{22} = \frac{1}{2}.$$

Note that this is not a whole number!

<u>Ex</u> A cts r.v. X has pdf

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)}, & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{4}{\pi}\int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{2}{\pi}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$u = 1+x^2$, $du = 2x\,dx$

$$= \frac{2}{\pi}\int_1^2 \frac{du}{u}$$

$$= \frac{2}{\pi}\Big[\ln|u|\Big]_1^2$$

$$= \frac{2}{\pi}(\ln 2 - \ln 1)$$

$$= \frac{2}{\pi}\ln 2$$

$$= \frac{\ln 4}{\pi} \approx .4413$$

Often we want to know the expectation of a fn $g(X)$ of a r.v. X (note that $g(X)$ is just another r.v.!).

<u>Thm 4.1</u> If X is a discrete r.v. with pdf $f(x)$, then the expected value of $g(X)$ is given by

$$E(g(X)) = \sum_{x} g(x) \cdot f(x).$$

If X is a cts r.v. with density $f(x)$, then the expected value of $g(X)$ is given by

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx.$$

Pf. Will only do this in the simplest case when X is discrete and takes on ftely many values and g is only finite-to-one.

Suppose that $g(x)$ takes on the value g_i when $x = x_{i1}, x_{i2}, \ldots, x_{in_i}$

Then

$$P(g(X) = g_i) = f(x_{i1}) + \cdots + f(x_{in_i})$$
$$= \sum_{j=1}^{n_i} f(x_{ij})$$

and if $g(X)$ takes on the values $g_1, g_2, \ldots, g_m$, then

$$E(g(X)) = \sum_{i=1}^{m} g_i P(g(X) = g_i)$$
$$= \sum_{i=1}^{m} g_i \sum_{j=1}^{n_i} f(x_{ij})$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n_i} g_i f(x_{ij})$$

$$= \sum_{x} g(x) \cdot f(x)$$

where this summation is over all possible values of X.

Ex. If X is the number of points rolled with a balanced die, then if $g(X) = 2X^2 + 1$

$$E(g(X)) = \sum_{x=1}^{6} (2x^2+1) \cdot \frac{1}{6}$$

$$= (2 \cdot 1^2 + 1) \cdot \frac{1}{6} + \cdots + (2 \cdot 6^2 + 1) \cdot \frac{1}{6}$$

$$= \frac{94}{3}.$$

Ex. If X is a cts r.v. with pdf

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

and $g(X) = e^{3X/4}$, find $E(g(X))$.

Sol^n. By Thm 4.1,

$$E(e^{3X/4}) = \int_0^\infty e^{3x/4} e^{-x}\,dx$$

$$= \int_0^\infty e^{-x/4}\,dx$$

$$= \lim_{M\to\infty} \left[\int_0^M e^{-x/4}\,dx \right]$$

$$= \lim_{M\to\infty} \left[-4e^{-x/4} \right]_0^M$$

$$= \lim_{M\to\infty} \left(-4e^{-M/4} - (-4) \right) = 4.$$

Useful facts.

<u>Thm 6.2</u> If a, b are constants, then

$$E(aX+b) = a\,E(X) + b.$$

<u>Pf.</u> Discrete case – see book for cts case.

Let $g(x) = ax+b$. Then by Thm 6.1

$$E(aX+b) = \sum_x (ax+b) f(x)\,dx$$

$$= a\sum_x x f(x)\,dx + \underbrace{b\sum_x f(x)\,dx}$$

$$= a\,E(X) + b \quad \blacksquare$$

$\| \; 1$ as f is a pdf!

<u>Cor 6.1</u> If a is a constant, then

$$E(aX) = a\,E(X)$$

<u>Cor 6.2</u> If b is a constant, then

$$E(b) = b$$

(here b is regarded as a constant r.v. which has only 1 value)

<u>Thm 4.3</u> If $c_1, \_\_, c_n$ are constants, then

$$E\left[\sum_{i=1}^{n} c_i g_i(X)\right] = \sum_{i=1}^{n} c_i E(g_i(X))$$

<u>Pf</u>. Cts case - see book for discrete

By Thm 4.1, if $g(X) = \sum_{i=1}^{n} g_i(X)$,

$$E\left[\sum_{i=1}^{n} c_i g_i(X)\right] = E(g(X))$$

$$= \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} c_i g_i(x)\right) f(x) dx$$

$$= \sum_{i=1}^{n} c_i \left(\int_{-\infty}^{\infty} g_i(x) f(x) dx\right)$$

$$= \sum_{i=1}^{n} c_i \, E(g_i(X))$$ by Thm 4.1 (backwards)

Ex. If X is a cts. r.v. with density

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

a) show that

$$E(X^r) = \frac{2}{(r+1)(r+2)}, \quad r \neq -1, -2$$

b) use this to evaluate

$$E((2X+1)^2).$$

Sol^n.

a) $$E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f(x)\,dx$$

$$= \int_0^1 x^r \cdot 2(1-x)\,dx$$

$$= 2\int_0^1 (x^r - x^{r+1})\,dx$$

$$= 2\left(\frac{1}{r+1} - \frac{1}{r+2}\right) = \frac{2}{(r+1)(r+2)}.$$

b) $E\left((2X+1)^2\right) = E\left(4X^2 + 4X + 1\right)$

$$= 4\underbrace{E(X^2)}_{r=2} + 4\underbrace{E(X)}_{r=1} + 1$$

$$= 4 \cdot \frac{2}{(2+1)(2+2)} + 4 \cdot \frac{2}{(1+1)(1+2)} + 1$$

by a) with $r=2$, $r=1$

$$= \frac{8}{12} + \frac{8}{6} + 1 = 3.$$

Ex. Show that

$$E((aX+b)^n) = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$$

Soln. Since $(ax+b)^n = \sum_{i=0}^{n} \binom{n}{i} (ax)^{n-i} b^i$,

$$E((aX+b)^n) = E\left[\sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i X^{n-i}\right]$$

$$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i}).$$

Can also extend expectation to the situation of one random variable which is a fn of two or more other r.v.'s which have a joint density.

Thm 4.4 If X & Y are discre r.v.'s with joint pdf $f(x,y)$, the expected value of $g(X,Y)$ (where g is a fn from $\mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g(X,Y)$ a r.v.) is

$$E(g(X,Y)) = \sum_x \sum_y g(x,y)\cdot f(x,y).$$

If X, Y are cts with joint density $f(x,y)$, the expected value of $g(X,Y)$ is

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\cdot f(x,y)\,dx\,dy.$$

Similar statements exist for fns of three or more variables.

Ex. For the tablets (yet again), find $E(X+Y)$.

Recall X, Y = # aspirin, sedative resp when 2 tablets are taken from a bottle containing 3 aspirin, 2 sedative and 4 laxative tablets. X & Y had joint pdf.

$$\begin{array}{cc|ccc} & & & x & \\ & & 0 & 1 & 2 \\ \hline & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{12} \\ y & 1 & \frac{2}{9} & \frac{1}{6} & \\ & 2 & \frac{1}{36} & & \end{array}$$

Then $$E(X+Y) = \sum_{x=0}^{2}\sum_{y=0}^{2}(x+y)\,f(x,y)$$

$$= (0+0)\cdot\frac{1}{6} + (0+1)\cdot\frac{2}{9} + (0+2)\cdot\frac{1}{36}$$

$$+ (1+0)\cdot\frac{1}{3} + (1+1)\cdot\frac{1}{6} + (2+0)\cdot\frac{1}{12}$$

$$= \frac{10}{9}.$$

Ex. If X, Y are cts with joint pdf

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y), & 0<x<1,\ 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$

find the expected value of $g(X,Y) = X/Y^3$.

Soln. $$E(X/Y^3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x}{y^3} \cdot f(x,y)\,dx\,dy$$

$$= \int_1^2\int_0^1 \frac{x}{y^3} \cdot \frac{2(x+2y)}{7}\,dx\,dy$$

$$= \frac{2}{7}\int_1^2\int_0^1 \left(\frac{x^2}{y^3} + \frac{2x}{y^2}\right)dx\,dy$$

$$= \frac{2}{7}\int_1^2 \left(\frac{1}{3y^3} + \frac{1}{y^2}\right)dy$$

$$= \frac{15}{84}.$$

A final useful result.

<u>Thm 4.5</u> If $c_1, \ldots, c_n$ are consts., then

$$E\left[\sum_{i=1}^{n} c_i g_i(X_1, \ldots, X_k)\right] = \sum_{i=1}^{n} c_i E(g_i(X_1, \ldots, X_k)).$$