§ 3.5 Multivariate Distributions

One can define many different r.v.'s over the same sample space.

e.g. Roll two dice, one red, one green. One r.v. might be the product of the numbers displayed, another their sum, another the sine of their difference, and so on.

We consider <u>multivariate distributions</u> where we have several r.v.'s defined on the same sample space and we are interested in all of their values simultaneously.

We shall first look at the <u>bivariate case</u> with two r.v.'s defined on the same sample space. (with just one r.v., we have the <u>univariate case</u>).

Ex Two tablets are selected at random from a bottle which contains 3 aspirin, 2 sedative, and 4 Laxaitive tablets. If X & Y are resp. the numbers of aspirin and sedative tablets included among the two drawn from the bottle, find the probs. associated with all possible values of X & Y.

Soln. Possible pairs are

$(0,0), (0,1), (1,0), (1,1), (0,2), (2,0)$.

To find the prob. for $(1,0)$, for example, we are drawing 1 aspirin 0 sedative & 1 laxitive tablets.

Aspirin can be drawn in $\binom{3}{1}$ ways

Sedative — — $\binom{2}{0}$ ways

Laxative — — $\binom{4}{1}$ ways

This gives a total of

$$\binom{3}{1}\binom{2}{0}\binom{4}{1} = 12$$

ways of drawing 1 aspirin & 1 laxative.

The total # of ways of drawing 2 tablets from the bottle (which contains 9 tablets) is

$$\binom{9}{2} = 36$$

and since our selection is random, they are all equally likely.

Thus by Thm 2-2 $\left(\frac{\#\text{ favourable}}{\#\text{ possible}}\right)$, the prob. of drawing 1 aspirin and 1 laxative is

$$\frac{\binom{3}{1}\binom{2}{0}\binom{4}{1}}{\binom{9}{2}} = \frac{12}{36} = \frac{1}{3}.$$

Continuing in this way, we get the following table

		x		
		0	1	2
	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
y	1	$\frac{2}{9}$	$\frac{1}{6}$	
	2	$\frac{1}{36}$		

In fact, it is not too hard to see that we have the following formula

$$P(X=x, Y=y) = \frac{\binom{3}{x}\binom{2}{y}\binom{4}{2-x-y}}{\binom{9}{2}}, \quad \begin{array}{l} 0 \le x, y \le 2 \\ 0 \le x+y \le 2. \end{array}$$

__Defn.__ If X, Y are discrete r.v.'s, the fn $f(x,y)$ given by

$$f(x,y) = P(X=x, Y=y)$$

for each pair of values (x,y) in the range of X, Y is called the __joint probability distribution__ of X and Y.

Like in Thm 3·1 (for one r.v.), it follows from the postulates of prob. that

__Thm 3·7__ A bivariate fn can serve as the joint pdf. of a pair of discrete r.v.'s X, Y iff its values $f(x,y)$ satisfy

1) $f(x,y) \geq 0$ for each pair of values (x,y) in its domain;

2) $\sum_x \sum_y f(x,y) = 1$, the double sum being over all the values in the domain of f.

Ex. Find the value of k for which

$$f(x,y) = kxy, \quad x, y = 1, 2, 3.$$

can serve as a joint pdf.

Soln. $f(x,y) \geq 0$ for each x, y with $1 \leq x, y \leq 3$, so condition 1 is satisfied automatically.

To check condition 2, want

$$f(1,1) + f(1,2) + f(1,3) + f(2,1) + f(2,2) + f(2,3)$$
$$+ f(3,1) + f(3,2) + f(3,3) = 1$$

ie $k + 2k + 3k + 2k + 4k + 6k$
$+ 3k + 6k + 9k = 1$

ie $36k = 1$,

so $k = \dfrac{1}{36}$.

<u>Def^n.</u> If X, Y are discrete r.v.'s, the fn F given by

$$F(x,y) = P(X \le x, Y \le y) = \sum_{s \le x} \sum_{t \le y} f(s,t)$$

for $-\infty < x < \infty$
$-\infty < y < \infty$

where $f(s,t)$ is the joint pdf of X and Y at (s,t) is called the <u>joint cumulative distribution function</u> of X and Y.

<u>Thm</u> If $F(x,y)$ is the value of the joint CDF of two discrete r.v.'s X & Y at (x,y), then

a) $\lim_{\substack{x \to -\infty \\ y \to -\infty}} F(x,y) = 0$, b) $\lim_{\substack{x \to \infty \\ y \to \infty}} F(x,y) = 1$

c) if $a<b$ and $c<d$, then $F(a,c) \le F(b,d)$.

Ex. For the tablets example, find $F(1,1)$, $F(-2,1)$, $F(3.7, 4.5)$

Sol'n.

$$F(1,1) = P(X \leq 1, Y \leq 1)$$

$$= f(0,0) + f(0,1) + f(1,0) + f(1,1)$$

$$= \frac{1}{6} + \frac{2}{9} + \frac{1}{3} + \frac{1}{6}$$

$$= \frac{8}{9}.$$

$$F(-2,1) = P(X \leq -2, Y \leq 1) = 0$$

$$F(3.7, 4.5) = P(X \leq 3.7, Y \leq 4.5) = 0$$

Note that as in the univariate case, the CDF is defined for all real numbers not just the values assumed by X & Y.