2.4 The Postulates of Probability and the Probability of an Event

For a sample space S and an event B, let $P(B)$ denote the probability of B. We have the following 3 postulates

1. $P(B) \geq 0$ for all events $B \subset S$.

2. $P(S) = 1$.

3. If $A_1, A_2, \ldots$ is a finite or countably infinite sequence of pairwise disjoint (mutually exclusive) events, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

or

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i).$$

Ex. An experiment has 4 possible outcomes A, B, C, D which are mutually exclusive. Explain why the following assignments of probability are not permissible.

a) $P(A) = .12$, $P(B) = .63$, $P(C) = .45$, $P(D) = -.2$

b) $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{3}$, $P(D) = \frac{1}{4}$.

Thm 2.1 If A is an event in a discrete sample space, then $P(A)$ equals the sum of the probabilities of the individual outcomes comprising A.

Note: A discrete sample space always contains countably many points (technical).

Pf: Let $O_1, O_2, O_3, \_\_\_\_$, be the outcomes (sample pts). which comprise A.

Then by postulate 3, since the O_i's are pairwise disjoint

$$P(A) = P(\bigcup_i O_i) = \sum_i P(O_i). \qquad \square$$

Ex. A die is biased so that each odd number is twice as likely to appear as each even number. Find $P(G)$ where G is the event that a number greater than 3 occurs on a single roll of the die.

Soln

$$S = \{1, 2, 3, 4, 5, 6\}.$$

If each even no. has prob w & each odd no. then has prob $2w$, we have

$$2w + w + 2w + w + 2w + w = 1$$

(postulate 2)

$$9w = 1$$

$$w = \frac{1}{9}$$

So 2, 4, 6 each have prob $\frac{1}{9}$ and
1, 3, 5 — — — — — — $\frac{2}{9}$.

Now

$$G = \{4, 5, 6\}$$

and by Thm 2-1

$$P(G) = P(4) + P(5) + P(6)$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{1}{9}$$

$$= \frac{4}{9}.$$

Ex. Flip a balanced coin twice.

What is the prob. of getting at least one head.

Sample space: $S = \{HH, HT, TH, TT\}$.

Coin is balanced, so each outcome has prob. $\frac{1}{4}$.

Event that I get at least one head is given by

$$A = \{HH, HT, TH\}$$

and by the previous thm

$$P(A) = P(HH) + P(HT) + P(TH)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{3}{4}$$

Ex. If, for a given experiment

$O_1, O_2, O_3, \ldots$ is a countably infinite sequence of (distinct) outcomes verify that

$$P(O_i) = \left(\frac{1}{2}\right)^i \quad \text{for } i = 1, 2, 3, \ldots$$

gives a probability measure.

Postulates 1, 3 follow from Thm 2.1, so we just need to check postulate 2, ie. $P(S) = 1$.

But $P(S) = P\left(\bigcup_{i=1}^{\infty} O_i\right) = \sum_{i=1}^{\infty} P(O_i)$ by Thm 2.1

$$= \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

<u>Thm 2.2</u> If an experiment can result in any one of N equally likely distinct outcomes and if n of these outcomes constitute an event A, then

$$P(A) = \frac{n}{N} \quad \left(\frac{\#\text{ favourable}}{\#\text{ possible}} \right).$$

<u>Pf.</u> Let $O_1, \ldots, O_n$ be the outcomes which comprise A. Then $A = O_1 \cup \cdots \cup O_n$ and $P(O_i) = \frac{1}{N}$ for each $1 \leq i \leq n$, so

$$P(A) = P(O_1 \cup \cdots \cup O_n)$$

$$= \sum_{i=1}^{n} P(O_i)$$ by Postulate 3 as the O_i are pairwise disjoint.

$$= \sum_{i=1}^{n} \frac{1}{N}$$

$$= \frac{n}{N}. \quad \square$$

Ex. What is the prob. of getting a full house (3 of a kind and a pair) in a 5 card poker hand dealt from a deck of 52 cards?

Soln. For a particular full house, (e.g. 3 kings, 2 aces) there are

$$\binom{4}{3}\binom{4}{2}$$

ways this can happen, while there are ${}_{13}P_2 = 13 \times 12$ possible types of card for a full house.

The total number of full houses is then

$13 \times 12 \binom{4}{3}\binom{4}{2}$ while the total number of poker hands is $\binom{52}{5}$.

The probability of a full house is then by Thm 2.2

$$\frac{\#\text{ favourable}}{\#\text{ possible}} = \frac{13 \times 12 \binom{4}{3}\binom{4}{2}}{\binom{52}{5}} \approx 0.0014.$$