§ 7.5 Solutions of Linear Systems: Existence, Uniqueness

Why rank is so very important ...

Theorem 1 Fundamental Theorem for Linear Systems

a) Existence A linear system of m eqns in n unknowns $x_1, \ldots, x_n$

$$
(1) \quad
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

is consistent (i.e. has a sol^n) iff the coefficient matrix A & the augmented matrix $\tilde{A}$ have the same rank. Here

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}, \quad
\tilde{A} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]
$$

b) <u>Uniqueness</u> The system (1) has precisely one sol^{n} iff this common rank r of A & $\tilde{A}$ equals n.

c) <u>Infinitely Many Solutions</u>

If $r<n$, (1) has ∞ many sol^{n}s.
-All sol^{n}s of the linear system are given by $n-r$ suitably chosen unknowns (the free variables) determining the remaining r unknowns (the basic variables).

d) <u>Gaussian Elimination (§7.3)</u>

If sol^{n}s exist, they can all be obtained by Gaussian elimination.

<u>Pf.</u> For d), we already saw in §7.3 that we can always do Gaussian elimination, or equivalently, that we can always convert the augmented matrix to row echelon form or reduced echelon form.

a) Recall the following two facts:

Elementary row operations do not change the solutions (if any) of a linear system.

Elementary row operations do not change either the row rank or the column rank of a matrix.

Hence it suffices to look at systems which are already in echelon form.

e.g.

$$\tilde{A} = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\bigcirc$ = pivot.

Here we have 4 pivots for $\tilde{A}$, but only 3 for A (block out last column to get A).
This system is inconsistent (gives $0 = 4$ as one equation!).

In general,

Solution exists (consistent)

$\Updownarrow$

No pivot in rightmost column of $\tilde{A}$

$\Updownarrow$

Same no. of pivot columns in A as in $\tilde{A}$

$\Updownarrow$

Col rank A = col rank $\tilde{A}$ (recall: col rank B = # pivot columns)

$\Updownarrow$

rank A = rank $\tilde{A}$

b) (Sketch) Recall in §7.3 how when we obtained the r.e.f., we also got the <u>parametric</u> <u>vector</u> <u>form</u> of the general sol$^{\text{n}}$ of a linear system. This showed how the sol$^{\text{n}}$ was determined by the values of the free variables.

e.g. suppose we had 6 unknowns $x_1, x_2, x_3, x_4, x_5, x_6$ and we find out from the r.e.f. that x_1, x_3, x_4 are basic and x_2, x_5, x_6 are free. The parametric vector form of the sol$^{\text{n}}$ then might look something like.

$$X = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -1 \\ 10 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -7 \\ 0 \\ 3 \\ 4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 7 \\ 1 \end{bmatrix}$$

The (slightly sketchy!) point here is that the vectors one obtains for the parametric vector form which are associated with the free variables are automatically <u>linearly independent</u>.

This has the consequence that choosing distinct sets of values for the free variables will give us distinct sol$^{\text{ns}}$.

Thus if we want our system to have just one unique sol$^{\Delta}$, there must be <u>no</u> free variables.

Hence

System is consistent and has a unique sol$^{\Delta}$

$$\Updownarrow$$

A & $\tilde{A}$ have the same rank (by a))
and there are no free variables

$$\Updownarrow$$

A & $\tilde{A}$ have the same rank
and there are no non-pivot colums of A (in echelon form).

$$\Updownarrow$$

A & $\tilde{A}$ have the same rank
and all columns of A are pivot

$$\Updownarrow$$

A & $\tilde{A}$ have the same rank
and this rank is n.

recall rank A = # pivot cols.
and A has n columns

A typical system of this type which has just one sol^n would have an augmented matrix which looked like

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 5 & 1 & 3 & 7 \\ 0 & 0 & 2 & 1 & -11 \\ 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

n.b. very easy to solve by back substitution.

d) (sketch only).

Recall that the r.e.f. gave us the parametric vector form of the sol^n for a linear system.

Here system is consistent and has infinitely many sol^ns.

$\Updownarrow$

A & $\tilde{A}$ have the same rank and there are free variables

$\Updownarrow$

A & $\tilde{A}$ have the same rank and not all columns of A are pivot

$\Updownarrow$

A & $\tilde{A}$ have the same rank and this rank r is $< n$ (the # of cols. of A).

In this case we also see from the parametric vector form how all sol'ns are given in terms of the free variables. With this, the proof of the thm. is complete. $\blacksquare$

Homogeneous Linear Systems

Recall from §7.3 that a homogeneous linear system is one where the r.h.s (i.e all the b_i's) is 0. Otherwise the b_i's are not all 0 and the system is called <u>inhomogeneous</u> (or even nonhomogeneous).

<u>Thm 2</u>. A homogeneous linear system

$$(4)\quad \begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

always has the <u>trivial solution</u> $x_1 = x_2 = \cdots = x_n = 0$.

Non-trivial solns exist iff rank $A < n$.

If rank $A = r < n$, these solns together with $x = \underline{0}$, form a vector space of dimension $n-r$, called the <u>solution space</u>.

In particular if x_1 & x_2 are soln vectors of (4), then $x = c_1 x_1 + c_2 x_2$ is also a soln for any choice of scalars c_1, c_2.

(n.b. this <u>does not hold</u> for inhomogeneous systems.)

<u>Pf</u>. That $x = 0$ is always a soln is obvious.

Note that it agrees with the fact that $\underline{b} = \underline{0}$ implies that rank $A =$ rank $\tilde{A}$ & so the system must be consistent by part a) of Thm 1.

If rank $A = r < n$, let x_1, x_2 be two solns and c_1, c_2 be any two scalars. Then $Ax_1 = Ax_2 = \underline{0}$ and so

$$A(c_1 x_1 + c_2 x_2) = c_1 A x_1 + c_2 A x_2 = c_1 \underline{0} + c_2 \underline{0} = \underline{0}.$$

(just matrix mult.)

Thus the solns form a vector space, as desired.

Finally, if rank $A = r < n$, recall the parametric vector form of the soln of (4). Because we have a homog. system where the rhs is $\underline{0}$, (something which is preserved by row ops), the only vectors in the parametric vector form of the soln are those associated with the free variables (the 'extra' vector comes from the rhs., but this is just $\underline{0}$ as we have already seen).

As the vectors in the parametric vector form are then all lin. ind

$$\begin{array}{c} \text{dimension of the space of solns of system (4)} \\ \| \\ \#\ \text{free variables} \\ \| \\ \#\ \text{non-pivot columns} \\ \| \\ n - \#\ \text{pivot columns} \\ \| \\ n - r \quad (\text{where } r = \text{rank } A). \end{array}$$

∎

<u>Corollary</u> For an $m \times n$ matrix A,

$$\text{rank } A + \text{nullity } A = n$$

Pf. If rank $A = r$, then nullity $A = \dim(\text{Nul } A) = n - r$, so

$$\text{rank } A + \text{nullity } A = r + (n-r) = n. \quad \blacksquare$$

<u>Thm 3</u> Homogeneous Linear Systems with fewer Eqns than Unknowns.

A homog. linear system with fewer eqns than unknowns always has nontrivial solns.

Pf. If we have m eqns & n unknowns, then $m < n$ and if A is the corresponding matrix, then

rank $A \leq m < n$.

Now apply Thm 2.

Inhomogeneous Linear Systems

Thm 4 If a inhomog linear system

$$(1) \quad \begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is consistent, then all of its sol$^{\text{n}}$s are obtained as

$$x = x_0 + x_h$$

where x_0 is any (fixed) sol$^{\text{n}}$ of (1) and x_h is the general sol$^{\text{n}}$ of the associated homogeneous system

$$(4) \quad \begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ \vdots \qquad\qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Pf. Let x_0 be a particular sol^{n} of (1) and suppose x_h is any sol^{n} of the homog system (4). Then, if we set $x = x_0 + x_h$,

$$Ax = A(x_0 + x_h) = Ax_0 + Ax_h = \underline{b} + \underline{0} = \underline{b},$$

so x is also a sol^{n}.

Conversely, suppose x is any sol^{n} of (4). Then if we set $x_h = x - x_0$,

$$Ax_h = A(x - x_0) = Ax - Ax_0 = \underline{b} - \underline{b} = \underline{0}$$

and so x_h is a sol^{n} of the homog. system (4).

Then we can write

$$\begin{aligned} x &= x_0 + (x - x_0) \\ &= x_0 + x_h \end{aligned}$$

where x_0 solves (1) & x_h solves (4).
In view of this, we are done. $\blacksquare$