§ 7.4 Linear Independence, Rank of a Matrix, Vector Spaces

Saw in the last section that a linear system can have

- no solution
- just one solution
- infinitely many solutions.

Motivates the idea of considering all (possible) solutions as one object and asking how 'big' this object is.

To explain just what 'big' means, we need the concepts of linear independence of vectors and the rank of a matrix.

Defns. Let $S = \{v_1, \ldots, v_m\}$ be a set of m vectors (row or column) with the same number of components. A linear combination of these vectors is an expression of the form

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m . \qquad (c_1, \ldots, c_m \text{ scalars})$$

(Note that we can add vectors and multiply vectors by scalars to get more vectors and so a linear combination will just be another vector).

The set of all possible linear combinations of vectors in S is called the <u>span</u> of S, written $\mathrm{span}(S)$ or $\mathrm{span}\{v_1, \ldots, v_m\}$.

Suppose now we ask, given a set $S = \{v_1, \ldots, v_m\}$, which choices of the scalars $c_1, \ldots, c_m$ give a linear combination which adds up to the zero vector. i.e. for which choices of $c_1, \ldots, c_m$ do we have

$$c_1 v_1 + \cdots + c_m v_m = \underline{0} \quad ?$$

Note that we always have at least one choice available, namely $c_1 = c_2 = \cdots = c_m = 0$.

We say S is <u>linearly</u> <u>independent</u> (lin. ind.) if this is the only choice, i.e. if

$$c_1 v_1 + \cdots + c_m v_m = \underline{0},$$

then we must have $c_1 = c_2 = \cdots = c_m = 0$.

If S is not linearly independent, then we say S is <u>linearly</u> <u>dependent</u> (lin dep.).
This means that here we can find scalars $c_1, \ldots, c_m$ not all of which are 0, for which

$$c_1 v_1 + \cdots + c_m v_m = 0.$$

(Note some of the scalars might be 0, but not all of them can be).

Suppose now, that c_3 say is one of the non-zero ones.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + \cdots + c_m v_m = 0$$

So

$$c_3 v_3 = -c_1 v_1 - c_2 v_2 - c_4 v_4 - \cdots - c_m v_m$$

$$v_3 = -\frac{c_1}{c_3} v_1 - \frac{c_2}{c_3} v_2 - \frac{c_4}{c_3} v_4 - \cdots - \frac{c_m}{c_2} v_m$$

i.e. v_3 is a lin. comb. of the other vectors in S.

Can keep doing this until we eliminate all the dependency in S (without making the span smaller).

Thm (Spanning Set Theorem)

Given a set S containing m vectors, we can find a subset S' of S which contains $r \leq m$ vectors which is lin. ind. and with $\text{span}(S') = \text{span}(S)$.

The set S' is an example of a basis. More on this later.

Ex. The set S of three vectors

$$a_1 = [3 \quad 0 \quad 2 \quad 2]$$
$$a_2 = [-6 \quad 42 \quad 24 \quad 54]$$
$$a_3 = [21 \quad -21 \quad 0 \quad -15]$$

is lin dep. because

$$6a_1 - \frac{1}{2}a_2 - a_3 = 0 \qquad \text{(check)}.$$

However, one can also check that

$S' = \{a_1, a_2\}$ is lin ind.

<u>Useful facts</u>

1. A set $S = \{v\}$ with only one vecter is lin ind. iff $v \neq \underline{0}$.

2. A set $S = \{v_1, v_2\}$ with only two vecters is lin ind. iff neither vecter is a scalar multiple of the other.

3. A set S which contains the zero vecter is automatically linearly dependent.

In the last example using fact 2., neither a_1 or a_2 is a scalar multiple of the other and so S' is lin. ind.

In fact $\text{span}(S) = \text{span}(S')$

which illustrates the spanning set theorem in this case.

Ex. Let $S = \{v_1, v_2, v_3, v_4\}$ where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Then S is lin ind. To see this, suppose

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \underline{0}.$$

ie
$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ c_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_3 \\ c_3 \\ c_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -c_4 \\ 0 \\ c_4 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 2c_2 + c_3 - c_4 \\ c_2 + c_3 \\ c_3 + c_4 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So

$$c_1 + 2c_2 + c_3 - c_4 = 0 \quad ①$$

$$c_2 + c_3 = 0 \quad ②$$

$$c_3 + c_4 = 0 \quad ③$$

$$c_4 = 0 \quad ④$$

ie. we have a linear system in upper triangular form which we can solve by back substitution.

④ $\Rightarrow$ $c_4 = 0$

sub $c_4 = 0$ in ③ $\Rightarrow$ $c_3 = 0$

sub. $c_3, c_4 = 0$ in ② $\Rightarrow$ $c_2 = 0$

sub. $c_2, c_3, c_4 = 0$ in ① $\Rightarrow$ $c_1 = 0$.

Hence $c_1 = c_2 = c_3 = c_4 = 0$

and so $S = \{v_1, v_2, v_3, v_4\}$ is lin ind.

Ex. Let $S = \{v_1, v_2, v_3, v_4\}$. where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then S is lin ind. To see this, suppose

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \underline{0}.$$

ie
$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $c_1 = c_2 = c_3 = c_4 = 0$ and so we have lin ind.

These vectors are known as the standard basis for $\mathbb{R}^4$. More on this later.

The last example was especially easy. To find the linear dependence or independence of more general sets of vectors, we need more machinery, particularly the notion of the rank of a matrix.

<u>Defn</u>. The <u>rank</u> of a matrix A is the maximum number of linearly independent row vectors of A. It is denoted by rank A.

<u>Ex</u>.

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Already saw that rank $A = 2$

Note. For a general matrix A, rank $A = 0$ iff $A = 0$.

Call two matrices A_1 and A_2 are <u>row equivalent</u> if A_2 can be obtained from A_1 using (finitely many) elementary row operations. Note: we can always reverse these operations and get A_1 back from A_2.

<u>Fact</u>: Elementary row operations do not change the rank of a matrix.

<u>Consequence</u>

<u>Thm 1</u>. Row equivalent matrices have the same rank.

This is useful! To find the rank of a matrix A, we apply row ops. to get the row echelon form where it is easy to see the rank.

Ex.

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$R2 + 2R1$, $R3 - 7R1$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

$R3 + \frac{1}{2}R2$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ -- row echelon form.

The rank of the echelon form is clearly 2, so rank$(A) = 2$.

This illustrates the following.

Thm 2 p vectors with n components are lin ind. if the matrix with these row vectors has rank p, but they are lin dep. if that matrix has rank $< p$.

We also have

Thm 3 Column Rank.

The rank r of a matrix also equals the max. no. of lin. ind. columns of A.

Hence A & A^T have the same rank.

Sketch of proof (see also in book).

We saw that elementary row ops. do not affect the (row) rank of a matrix.

It turns out that they also do not affect the column rank (basic reason for this is that row ops. preserve any dependence relations between the column vectors, even though they change the vectors themselves).

Hence to find both the row & column rank of A, it suffices to examine the row echelon form of A.

e.g. $$\begin{bmatrix} 0 & \bullet & \times & & \times & \times & \times & \times \\ 0 & 0 & \bullet & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & \bullet & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\bullet$ = pivot

$\times$ = not a pivot

Assertion (take it or leave it!)

The pivot rows of the row echelon form give us a lin. ind set and all other rows are lin combs. of these rows.

The same thing holds for columns (of the reduced matrix, not A!).

Hence

row rank of A = # pivot rows

= # pivots

= # pivot columns

= column rank of A.

Combining Thms 2, 3 gives

<u>Theorem 4</u> Linear Dependence of Vectors.

A set S of p vectors with $n < p$ components is always lin dep.

Pf. Let A be the matrix with these vectors as rows.

Then A has p rows and $n < p$ cols.

Thus the col. rank of A is $\leq n < p$ and so by thm 3, the rank of A is $< p$.

Then by Thm 2, S is lin dep. ∎

Vector Spaces

<u>Defns.</u> A vector space V is a set of vectors s.t. for any $v, w \in V$ all possible linear combinations

$$\alpha v + \beta w$$

are also in V.

Note that in particular the sum

$$v + w \qquad (\alpha = \beta = 1)$$

is then in V, as is the scalar multiple

$$\alpha v \qquad (\beta = 0).$$

The max. no. of lin-ind. vectors in V is called the <u>dimension</u> of V. We shall work only with spaces where this number is finite.

A set of vectors S is called a <u>basis</u> of V if S is lin ind and span $(S) = V$.

<u>Fact</u> all bases of a (finite-dimensional) vector space V have the same number of elements and this number is the <u>dimension</u> of V.

A subspace of a vector space V is a (non-empty) subset of V which is a vector space in its own right.

<u>Ex.</u> The span of the three vectors $S = \{a_1, a_2, a_3\}$ from earlier is a 2-dim space and $S' = \{a_1, a_2\}$ is a basis.

Thm5 (and Ex) The vector space $\mathbb{R}^n$ consisting of all vectors with n (real) components is a vector space of dimension n.

Pf. $e_1 = [1, 0, 0, \ldots, 0]$

$e_2 = [0, 1, 0, \ldots, 0]$

$\vdots$

$e_n = [0, 0, \ldots, 1]$

is a basis (called the standard basis for $\mathbb{R}^n$).

Fact If S is a set of vectors in $\mathbb{R}^n$, then span (S) is a subset of $\mathbb{R}^n$.

Defn. The space spanned by the rows of a matrix A is called the row space of A, written row (A).

The space spanned by the columns of A is called the column space of A, written col (A).

Thm 3 then immediately tells us.

<u>Theorem 6</u> $\dim(\text{row } A) = \dim(\text{col } A) = \text{rank } A$.

Finally, the set of all x for which $Ax = 0$ is easily seen to be a v.s. It is called the <u>nullspace</u> of A and its dimension is referred to as the <u>nullity</u> of A.

<u>Fact</u>

$$\text{rank } A + \text{nullity } A = \#\text{ columns of } A.$$