§7.2 Matrix Multiplication

<u>Defn.</u> Let $A = (a_{ij})$ be $m \times n$ and
$B = (b_{jk})$ be $n \times p$.

The matrix product $C = AB$ is an $m \times p$ matrix $C = (c_{ik})_{\substack{1 \le i \le m \\ 1 \le k \le p}}$ whose entries are given by

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + \cdots + a_{in} b_{nk},$$

$$1 \le i \le m$$
$$1 \le k \le p.$$

<u>Note</u> The matrix product AB only makes sense provided B has the same number of rows as A has columns.

<u>Easy to remember</u>

$$\underset{(m \times n)}{A} \cdot \underset{(n \times p)}{B} = \underset{(m \times p)}{C}$$

Ex.

$$\begin{bmatrix} 2 & 1 & 7 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\times 1 + 1\times 3 + 7\times -1 \\ 4\times 1 + 3\times 3 + 1\times -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 12 \end{bmatrix}$$

$2 \times 3 \qquad 3 \times 1 \qquad 2 \times 1$

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}$$

Last example shows that the order of matrix multiplication matters (ie. it is not commutative). In general, $AB \neq BA$. Also $AB = 0$ does not imply that $A = 0$ or $B = 0$.

Rules for Matrix Multiplication

a) $(kA)B = k(AB) = A(kB)$ written kAB or AkB

b) $A(BC) = (AB)C$ written ABC

c) $(A+B)C = AC + BC$

d) $C(A+B) = CA + CB$

Here k is a scalar and A, B, C are sized so that all the matrix products make sense.

If $v = [v_1, \ldots, v_n]$, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

are $1\times n$ and $n\times 1$ row and column vectors (resp.), then vw is 1×1 (ie just a number).

$$vw = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Now if A is $m \times n$, B is $n \times p$

and if $a_i = [a_{i1}, \ldots, a_{in}]$, $b_k = \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix}$

denote the rows and columns of A, B resp.,
then we can write

$$AB = \begin{bmatrix} - & a_1 & - \\ - & a_2 & - \\ & \vdots & \\ - & a_m & - \end{bmatrix} \begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_p \\ a_2 b_1 & & & \vdots \\ \vdots & & & \\ a_m b_1 & & & a_m b_p \end{bmatrix}$$

where the products in the entries for AB are the same as the product of a row & a column vector we had earlier.

Ex.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 4, \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -3$$

$$\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 11, \quad \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -8$$

So $AB = \begin{bmatrix} 4 & -3 \\ 11 & -8 \end{bmatrix}$.

Linear Transformations and Matrix Multiplication

A linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ is a function of the type

$$y_1 = a_{11} x_1 + a_{12} x_2$$
$$y_2 = a_{21} x_1 + a_{22} x_2$$

Can write this in matrix notation as

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + a_{22} x_2 \end{bmatrix}$$

Suppose we also have another linear transformation

$$x_1 = b_{11} w_1 + b_{12} w_2$$
$$x_2 = b_{21} w_1 + b_{22} w_2$$

which gives x in terms of w.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Bw = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_{11} x_1 + b_{12} x_2 \\ b_{21} x_1 + b_{22} x_2 \end{bmatrix}$$

How to get y in terms of w?

$$y = Ax = A(Bw) = (AB)w$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

So to get y in terms of w, we use the matrix product AB.

MORAL

linear transformations $\longleftrightarrow$ matrices

composition of linear transformations $\longleftrightarrow$ matrix multiplication.

This works for spaces of any dimension (not just $\mathbb{R}^2$).

Transposition

Defn. If the matrix $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ is $m \times n$, then the transpose of A, A^T is the $n \times m$ matrix $(a_{ji})_{\substack{1 \le j \le n \\ 1 \le i \le m}}$. The columns of A become rows of A^T and the rows of A become columns of A^T. (Just swap the indices).

$$A = \underbrace{\left.\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}\right| m}_{n}, \quad A^T = \underbrace{\left.\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & \\ \vdots & & & \\ \vdots & & & \\ a_{1n} & \cdots & \cdots & a_{mn} \end{bmatrix}\right| n}_{m}$$

Ex.

$$A = \begin{bmatrix} 1 & 4 & 6 & 7 \\ 2 & 1 & 2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 6 & 2 \\ 7 & 1 \end{bmatrix}$$

Rules for Transposition

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(cA)^T = cA^T$

d) $(AB)^T = B^T A^T$. (Note reversed order!)

Special Matrices

Symmetric $A^T = A$ e.g. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Antisymmetric $A^T = -A$ e.g. $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

Both symmetric and antisymmetric matrices must be square. The diagonal entries of an antisymmetric matrix are always zero (why?).

Triangular Matrices

Upper Triangular - 0s below the diagonal

e.g. $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 7 & 2 \end{bmatrix}$

Lower Triangular - 0s above the diagonal

e.g. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}$

Diagonal - Square matrices with 0's off the diagonal

e.g. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Special case - I_n the $n \times n$ identity matrix.

$$I_n = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \bigcirc \\ & & 1 & \\ & \bigcirc & & \ddots & \\ & & & & 1 \end{bmatrix}}_{n} \Big\} n$$

If A is $m \times n$, then

$$I_m A = A = A I_n .$$

Convenient to work with the augmented or partitioned matrix $\tilde{A}$ for this system where

$$\tilde{A} = \underbrace{\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]}_{n+1} \Big\} m$$

Ex. 2 equations in 2 unknowns.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Each equation gives us a line in $\mathbb{R}^2$ and the sol^n (if any) of the system is determined by how (and if) these lines intersect.

$$2x_1 + 5x_2 = 2$$

$$0x_1 + 13x_2 = -26$$

$$\left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array}\right].$$

Row 2 + 2(Row 1).

We now have the same upper triangular system as before which we solve by back substitution like last time to get a unique sol$^{\underline{n}}$ $x_1 = 6$, $x_2 = -2$.