§4.2 Basic Theory of Systems of ODEs

We generalize some of the results of § 3.1 to systems of ODEs

(1)
$$y_1' = f_1(t, y_1, \ldots, y_n)$$
$$y_2' = f_2(t, y_1, \ldots, y_n)$$
$$\vdots$$
$$y_n' = f_n(t, y_1, \ldots, y_n)$$

If we let $\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $\underline{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$, we can write the system in vector form as

(1) $$\underline{y}' = \underline{f}(t, \underline{y})$$

If f does not depend explicitly on t, so that

$$\underline{y}' = \underline{f}(\underline{y}),$$

the system is called autonomous. Otherwise we say it is non-autonomous.

A solution of the system (1) on an open interval $a < t < b$ is a set of n differentiable fns.

$$y_1 = h_1(t),\ y_2 = h_2(t), \ldots, y_n = h_n(t)$$

on $a < t < b$ which satisfy (1) everywhere on this interval. In vector form if we let

$$\underline{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix},$$ then our soln is given by $\underline{y} = \underline{h}(t)$.

An <u>initial value problem</u> for the system (1) consists of (1) together with n given <u>initial conditions</u>

(2) $y_1(t_0) = K_1, \quad y_2(t_0) = K_2, \quad \ldots \quad y_n(t_0) = K_n$

or, in vector form

$$\underline{y}(t_0) = K, \qquad K = \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix}.$$

Here t_0 is some specified time in the interval $a < t < b$ under consideration.

For systems of ODEs, we have the following theorem which is of fundamental importance.

Theorem 1

For the system

$$y_1' = f_1(t, y_1, \ldots, y_n)$$

$$\vdots$$

$$y_n' = f_n(t, y_1, \ldots, y_n)$$

if the fns $f_1, \ldots, f_n$ are cts. and have cts partial derivatives $\frac{\partial f_1}{\partial y_1}, \ldots, \frac{\partial f_1}{\partial y_n}, \frac{\partial f_2}{\partial y_1}, \ldots, \frac{\partial f_n}{\partial y_n}$ in some domain R of $t y_1 y_2 \ldots y_n$-space, then if $(t_0, K_1, K_2, \ldots, K_n) \in R$, the IVP consisting of this system together with the ICs.

$$y_1(t_0) = K_1, \ldots, y_n(t_0) = K_n$$

will have a solution on some open interval $t_0 - \alpha < t < t_0 + \alpha$ $(\alpha > 0)$ containing t_0.

Furthermore, this solution is unique.

Linear Systems

A system of n ODEs is a <u>linear system</u> if it is linear in $y_1, \ldots, y_n$, i.e. it can be written

$$(3) \qquad \begin{aligned} y_1' &= a_{11}(t)\,y_1 + \cdots + a_{1n}(t)\,y_n + g_1(t) \\ &\;\;\vdots \\ y_n' &= a_{n1}(t)\,y_1 + \cdots + a_{nn}(t)\,y_n + g_n(t) \end{aligned}$$

In vector form, this becomes

$$\underline{y}' = A\underline{y} + g$$

where $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}$, $\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $g = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$.

If $g \equiv 0$, we have the <u>homogeneous</u> linear system

(4) $$\underline{y}' = A\underline{y}.$$

Otherwise we say the system is <u>inhomogeneous</u>.

Note that for a linear system, we have

$$\frac{\partial f_j}{\partial y_k} = a_{jk}(t), \quad \begin{matrix} 1 \le j \le n \\ 1 \le k \le n \end{matrix}$$

In this case Thm 1 can be stated as follows.

<u>Thm 2</u> Existence and Uniqueness in the Linear Case.

Let the fns $a_{jk}(t)$ and $g_j(t)$ all be cts fns of t on an open interval $\alpha < t < \beta$ containing $t = t_0$. Then (3) has a solⁿ $\underline{y}(t)$ on this interval which is unique.

As for a single homog linear ODE we have

Thm 3 Superposition / Linearity for Homog. Systems.

Any linear combination of solutions of the homog. linear system (4) is again a solution. In other words, the solutions of (4) form a vector space whose 'vectors' are functions

Pf. Similar to that of §2.1 Thm 1 (P. 47).
also §3.2 Thm 1 (P. 106).

E.g. for a lin comb $\underline{y} = c_1 \underline{y}^{(1)} + c_2 \underline{y}^{(2)}$ of two sol'ns $\underline{y}^{(1)}, \underline{y}^{(2)}$.

$$\underline{y}' = \left(c_1 \underline{y}^{(1)} + c_2 \underline{y}^{(2)}\right)'$$

$$= c_1 \underline{y}^{(1)\prime} + c_2 \underline{y}^{(2)\prime}$$

$$= c_1 A \underline{y}^{(1)} + c_2 A \underline{y}^{(2)}$$

$$= A(c_1 \underline{y}^{(1)} + c_2 \underline{y}^{(2)})$$

$$= A \underline{y}$$

Thus $\underline{y}' = A\underline{y}$ & so $\underline{y}$ is also a sol^{n}.

Basis, General Solution, Wronskian

A basis or fundamental system of sol^{n}s of a homog linear system of n eq^{n}s on some open interval J (in t) is a set of n sol^{n}s $\underline{y}^{(1)}(t), \ldots, \underline{y}^{(n)}(t)$ which are lin ind. on J. A linear combination

$$\underline{y} = c_1 \underline{y}^{(1)} + \cdots + c_n \underline{y}^{(n)}$$

with $c_1, \ldots, c_n$ arbitrary is called a <u>general solution</u> of the system. If we fix values for the consts. $c_1, \ldots, c_n$, we obtain a <u>particular solution</u>.

Can show that if the matrix entry fns $a_{jk}(t)$ are cts on J, then we can find a basis of solns.

If we put the n solns $\underline{y}^{(1)}, \ldots, \underline{y}^{(n)}$ together as the columns of a matrix

$$Y = [\underline{y}^{(1)} \cdots \underline{y}^{(n)}],$$

then the determinant of this matrix is the Wronskian of $\underline{y}^{(1)}, \ldots, \underline{y}^{(n)}$, $W(\underline{y}^{(1)}, \ldots, \underline{y}^{(n)})$.

ie

$$W(\underline{y}^{(1)}, \ldots, \underline{y}^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ \vdots & \vdots & & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{vmatrix}$$

These sol^ns form a basis on J iff W is not zero at any $t_1 \in J$.

Also either $W \equiv 0$ on J or W is nonzero everywhere on J.

(Similar to § 2.6, 3.1).

Finally, if $\underline{y} = c_1 \underline{y}^{(1)} + \quad + c_n \underline{y}^{(n)}$ is another sol^n, then we can express $\underline{y}$ as

$$\underline{y} = Y \underline{c} \quad \text{where} \quad \underline{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$