§3.3 Inhomogeneous ODEs

We consider ODEs of the type

(1) $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)$

where $r(x) \not\equiv 0$.

As in the second order case, a general sol^{n} of (1) on some open interval I is of the form

(2) $y(x) = y_h(x) + y_p(x)$,

Here $y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is a general sol^{n} of the corresponding homog ODE

(3) $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$

on I and $y_p(x)$ is any sol^{n} of (1) on I with no consts (ie a particular sol^{n}).

If $p_0(x), \ldots, p_{n-1}(x), r(x)$ are all cts on I, then (1) has a general sol$^{\Delta}$ on I.

In addition if we add n <u>initial conditions</u>

(4) $y(x_0) = K_0,\ y'(x_0) = K_1, \ldots, y^{(n-1)}(x_0) = K_{n-1}$,

for some $x_0 \in I$, then the associated IVP has a unique sol$^{\Delta}$ on I.

The proofs of all of these facts are similar to the 2nd order case.

Method of Undetermined Coefficients

We consider inhomog. ODEs with const. coeffs.

(5) $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x)$.

A particular sol^n $y_p(x)$ can be found using the method of undetermined coefficients. The table is the same as in § 2.7 and the rules are similar,

A) Basic Rule Same as in § 2.7

B) Modification Rule If a term in your choice for $y_p(x)$ is a sol^n of the homog eq^n (3), then multiply $y_p(x)$ by x^k where k is the smallest positive integer s.t. no term in $x^k y_p(x)$ is a sol^n of (3).

C) Sum Rule Same as in § 2.7.

Ex. IVP Modification Rule.

$$y''' + 3y'' + 3y' + y = 30e^{-x}$$

$$y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -47.$$

Sol^n. Step 1

Char eq^n is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

ie $(\lambda + 1)^3 = 0$

So $\lambda = -1$ is a triple root (and there are no other roots).

The general sol^n of the associated homog. eq^n is then

$$y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$$

$$= (c_1 + c_2 x + c_3 x^2) e^{-x}.$$

<u>Step 2.</u> If we tried $y_p = Ce^{-x}$ as our particular sol^n, we would get

$$-C + 3C - 3C + C = 30$$

ie $$0 = 30$$

which is impossible. $y_p = Cxe^{-x}$, $y_p = Cx^2e^{-x}$ lead to similar problems.

The modification rule calls for

$$y_p = Cx^3e^{-x}$$

Then $$y_p' = C(3x^2 - x^3)e^{-x}$$

$$y_p'' = C(6x - 6x^2 + x^3)e^{-x}$$

$$y_p''' = C(6 - 18x + 9x^2 - x^3)e^{-x}$$

If we subst all this into the ODE and cancel the common factor e^{-x}, we get

$$C\left(6-\cancel{18x}+\cancel{9x^2}-\cancel{x^3}\right)+3C\left(\cancel{6x}-\cancel{6x^2}+\cancel{x^3}\right)$$

$$+3C\left(\cancel{3x^2}-\cancel{x^3}\right)+\cancel{Cx^3}=30$$

The terms in x, x^2, x^3 are all 0 and we are left with

$$6C = 30$$

so $\quad C = 5 \quad$ and $\quad y_p = 5x^3e^{-x}.$

<u>Step3</u>. Now write $y = y_h + y_p$ and diff and apply the ICs.

$$y = y_h + y_p = (c_1 + c_2 x + c_3 x^2)e^{-x} + 5x^3e^{-x}$$

$$y(0) = c_1 = 3.$$

$$y' = \left(-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3\right)e^{-x}$$

$$y'(0) = -3 \Rightarrow -3 + c_2 = -3 \Rightarrow c_2 = 0.$$

$$y'' = \left(-3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3\right)e^{-x}$$

$$y''(0) = -47 \Rightarrow 3 + 2c_3 = -47 \Rightarrow c_3 = -25$$

Hence the sol^{n} of our IVP is

$$y = (3 - 25x^2)e^{-x} + 5x^3e^{-x}.$$

Variation of Parameters

The formula for y_p is

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)\, dx$$

$$= y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\, dx + \cdots + y_n(x) \int \frac{W_n(x)}{W(x)} r(x)\, dx$$

Here $W_k(x)$ is obtained for each $1 \le i \le n$ by calculating the determinant where we replace the kth column of $W(x)$

$$\begin{bmatrix} y_k(x) \\ \vdots \\ y_k^{(n-1)}(x) \end{bmatrix}$$

by the kth standard basis vector

$$e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow k\text{th position.}$$

Note that when $n=2$, we recover the earlier formula from § 2.10 as then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

EXAMPLE 2 Variation of Parameters. Nonhomogeneous Euler–Cauchy Equation

Solve the nonhomogeneous Euler–Cauchy equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \ln x \qquad (x > 0).$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** Substitution of $y = x^m$ and the derivatives into the homogeneous ODE and deletion of the factor x^m gives

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0.$$

The roots are 1, 2, 3 and give as a basis

$$y_1 = x, \qquad y_2 = x^2, \qquad y_3 = x^3.$$

Hence the corresponding general solution of the homogeneous ODE is

$$y_h = c_1x + c_2x^2 + c_3x^3.$$

Step 2. Determinants needed in **(7).** These are

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

Step 3. Integration. In (7) we also need the right side $r(x)$ of our ODE in standard form, obtained by division of the given equation by the coefficient x^3 of y'''; thus, $r(x) = (x^4 \ln x)/x^3 = x \ln x$. In (7) we have the simple quotients $W_1/W = x/2$, $W_2/W = -1$, $W_3/W = 1/(2x)$. Hence (7) becomes

$$\begin{aligned} y_p &= x \int \frac{x}{2}\, x \ln x \, dx - x^2 \int x \ln x \, dx + x^3 \int \frac{1}{2x}\, x \ln x \, dx \\ &= \frac{x}{2}\left(\frac{x^3}{3} \ln x - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2} \ln x - \frac{x^2}{4}\right) + \frac{x^3}{2}(x \ln x - x). \end{aligned}$$

Simplification gives $y_p = \frac{1}{6}x^4 (\ln x - \frac{11}{6})$. Hence the answer is

$$y = y_h + y_p = c_1x + c_2x^2 + c_3x^3 + \tfrac{1}{6}x^4 (\ln x - \tfrac{11}{6}).$$

Figure 74 shows y_p. Can you explain the shape of this curve? Its behavior near $x = 0$? The occurrence of a minimum? Its rapid increase? Why would the method of undetermined coefficients not have given the solution? ■