§ 3.2 Homogeneous Linear ODEs with Constant Coefficients

We look at nth order homog. ODEs with const. coeffs which we can write in the form

$$(\dagger) \quad y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

As in Chapter 2, if we let $y = e^{\lambda x}$, then

$$y^{(i)} = \lambda^i e^{\lambda x}$$

and if we substitute into $(\dagger)$ we get

$$(\dagger\dagger) \quad \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

which is the <u>characteristic equation</u> associated with the ODE.

<u>Fact</u>: For a polynomial equation

$$a_d x^d + a_{d-1} x^{d-1} + \text{———} + a_1 x + a_0 = 0$$

a given root is either

a) real

or

b) one of a complex conjugate pair (e.g. $-1 \pm 3i$).

As far as (†) is concerned, there are 4 possibilities for the roots of the char eq^n which we now discuss.

I. Distinct Real Roots

If all the n roots $\lambda_1, \ldots, \lambda_n$ of (††) are real and distinct, then if we let

$$y_1 = e^{\lambda_1 x}, \quad \ldots \quad , y_n = e^{\lambda_n x}$$

$$W(y_1, \ldots, y_n) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & & \lambda_n e^{\lambda_n x} \\ \vdots & & & \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \cdots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix}$$

$$= e^{\lambda_1 x} \cdots e^{\lambda_n x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & & \lambda_n \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \lambda_n^{n-1} \end{vmatrix}$$

$$= e^{(\lambda_1 + \cdots + \lambda_n) x} \prod_{j<k} (\lambda_j - \lambda_k)$$

(Vandermonde determinant)

As the exponential fn $e^{(\lambda_1 + \cdots + \lambda_n)x}$ can never have value 0, we see that $W(y_1, \ldots, y_n) \neq 0$ and $y_1, \ldots, y_n$ are thus lin ind (on $\mathbb{R}$) iff $\lambda_1, \ldots, \lambda_n$ are all distinct. We have proved

<u>Thm 1</u> Basis

Solutions $y_1 = e^{\lambda_1 x}, \ldots, y_n = e^{\lambda_n x}$ of (+) (with any real or complex λ_j's) form a basis of sol^{ns} of (+) on $\mathbb{R}$ (and thus any open interval) iff all the roots $\lambda_1, \ldots, \lambda_n$ of (++) are distinct.

A similar argument gives the following more general version.

<u>Thm 2</u> Any number of sol^{ns} of (+) of the form $e^{\lambda x}$ are lin ind on (any) open interval I iff the corresponding λ's are all distinct.

Ex. $y''' - 2y'' - y' + 2y = 0$

The char eqn is

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

One can find that the 3 roots are

$-1, 1, 2$ (e.g. find one by inspection and the other 2 by algebraic long division).

Since these are real and distinct, the general soln of the ODE is

$$y = C_1 e^{-x} + C_2 e^{x} + C_3 e^{2x}.$$

II Simple Complex Roots

As we said earlier any complex roots of the char eq^n (††) will be in complex conjugate pairs. Thus $\lambda = \gamma + i\omega$ is a root iff $\bar{\lambda} = \gamma - i\omega$ is also a root and as in Ch. 2, we get two lin ind real-valued sol^{ns}.

$$y_1 = e^{\gamma x}\cos(\omega x), \quad y_2 = e^{\gamma x}\sin \omega x.$$

<u>Ex.</u> Solve the IVP

$$y''' - y'' + 100y' - 100y = 0$$

$$y(0) = 4, \quad y'(0) = 11, \quad y''(0) = -299$$

<u>Sol^n.</u> Char eq^n is

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$$

One finds by inspection that 1 is a root and division by $\lambda - 1$ shows that the other roots are $\pm 10i$.

The general sol^n is then

$$y = c_1 e^x + A\cos(10x) + B\sin(10x)$$

To use the IC's to find c_1, A, B we need to diff twice.

$$y' = c_1 e^x - 10A\sin(10x) + 10B\cos(10x)$$

$$y'' = c_1 e^x - 10A\cos(10x) - 10B\sin(10x)$$

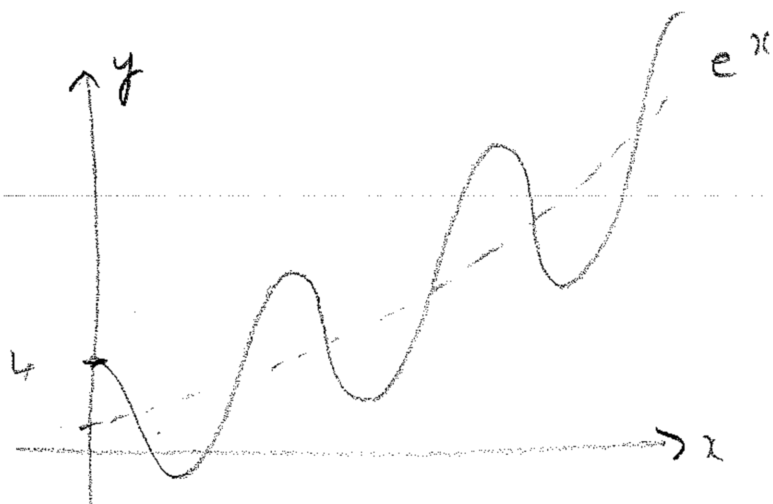
Substituting in the ICs at $x = 0$

$$\begin{aligned} c_1 + A \quad\quad &= 4 \\ c_1 \quad\quad + 10B &= 11 \\ c_1 - 100A \quad\quad &= -299. \end{aligned}$$

Solving this (e.g. using Gaussian elimination) gives $c_1 = 1$, $A = 3$, $B = 1$ and so the particular sol^{n} is

$$y = e^{x} + 3\cos(10x) + \sin(10x)$$

Sol^{n} looks like



III Multiple Real Roots

If a real double root occurs, say $\lambda_1 = \lambda_2$ then $y_1 = e^{\lambda_1 x} = e^{\lambda_2 x} = y_2$ and as in Ch. 2, we take y_1 and xy_1 as our corresponding lin ind sol^ns.

More generally, if λ is a real root of order m, then the m corresponding lin ind sol^ns are

$$e^{\lambda x}, \ x e^{\lambda x}, \ x^2 e^{\lambda x}, \ \ldots, \ x^{m-1} e^{\lambda x}$$

One can show that the Wronskian of these sol^ns is non-zero and that they are lin ind. (and also lin ind when we add the remaining sol^ns for the other roots of the char eq^n).

Ex. $y^{v} - 3y^{iv} + 3y''' - y'' = 0$

Char eqⁿ is

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

0 is a double root and

1 is a triple root.

The general solⁿ is then

$$y = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2) e^x.$$

IV Multiple Complex Roots

Proceed similarly as for multiple real roots.

Eg. If $\lambda = \gamma + i\omega$ is a complex double root, then so is the conjugate $\bar{\lambda} = \gamma - i\omega$ and the corresponding lin ind sol^ns are

$$e^{\gamma x}\cos\omega x,\ e^{\gamma x}\sin\omega x,\ xe^{\gamma x}\cos\omega x,\ xe^{\gamma x}\sin\omega x$$

and the general sol^n is

$$y = e^{\gamma x}\left[(A_1 + A_2 x)\cos\omega x + (B_1 + B_2 x)\sin\omega x\right]$$

(assuming there are no other roots),

For complex triple roots, one would obtain 2 more sol^ns $x^2 e^{\gamma x}\cos\omega x,\ x^2 e^{\gamma x}\sin\omega x$ and so on.