§ 2.6 Existence and Uniqueness of Solutions, the Wronskian

Suppose we have an IVP for a 2nd order linear homog. ODE

$$y'' + p(x)y' + q(x)y = 0$$

$$y(0) = k_0, \quad y'(0) = K_1.$$

Two important questions are

1) Does a solution exist? (Existence)

2) Is such a solution unique? (Uniqueness)

Thm 1 Existence and Uniqueness Theorem for IVPs

If $p(x)$ and $q(x)$ are cts on some open interval I and $x_0 \in I$ then the IVP

$$y'' + p(x)y' + q(x)y = 0$$

$$y(0) = K_0, \quad y'(0) = K_1$$

has a unique sol$^{\underline{n}}$ $y = y(x)$ on I.

Pf. Difficult!

Linear Independence of Solutions

Recall from § 2.1 that two fns y_1, y_2 defined on an open interval I are linearly independent if

$$c_1 y_1 + c_2 y_2 = 0 \quad \text{on } I \quad \text{implies}$$

$$c_1 = c_2 = 0.$$

Otherwise, y_1, y_2 are linearly dependent and we saw in this case that they were proportional, so that either

$$y_1 = k y_2 \quad \text{on } I \quad \text{for some const } k$$

or

$$y_2 = l y_1 \quad \text{on } I \quad \text{for some const. } l$$

(or both).

For two fns y_1, y_2 which are differentiable on I, their <u>Wronskian</u> $W(y_1, y_2)$ is defined by

$$W(y_1, y_2) = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

The importance of this is seen in

<u>Thm 2</u> Let the ODE

$$y'' + p(x) y' + q(x) y = 0$$

have cts coeffs on an open interval I. Then two sol$^{\underline{n}}$s y_1, y_2 of this ODE are lin. dep. on I iff their Wronskian $W(y_1, y_2)$ is 0 at some pt $x_0 \in I$. Furthermore if $W(y_1, y_2) = 0$ at some $x_0 \in I$, then $W(y_1, y_2) = 0$ <u>everywhere</u> on I. Hence if there is

an $x_1 \in I$ at which $W(y_1, y_2) \neq 0$,
then y_1, y_2 are lin ind on I.

Pf. a) Let y_1, y_2 be lin dep. on I.
Then either $y_1 = k y_2$ or $y_2 = l y_1$
on I (or both).

If $y_1 = k y_2$, then

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= k y_2 y_2' - y_2 k y_2'$$

$$\equiv 0 \quad \text{on } I$$

and if $y_2 = l y_1$, then a similar argument shows $W(y_1, y_2) \equiv 0$ on I in this case also.

Hence in this case $W(y_1, y_2)$ is zero everywhere on I and so if we let x_0 be any pt of I, then $W(y_1, y_2) = 0$ at x_0

b) Conversely suppose $W(y_1, y_2) = 0$ at some $x_0 \in I$.

Then

$$\det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} = 0$$

and so if we let Y be the matrix

$$Y = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix},$$

then $\det Y = 0$ and so $\operatorname{rank} Y < 2$.

Since $\operatorname{rank} Y + \operatorname{nullity} Y = 2$,

this means that $\operatorname{nullity} Y \geq 1$ and

so we can find a <u>non-zero</u> vector

$$v = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{with} \quad Yv = \underline{0}$$

Then

$$Yv = \underline{0}$$

$$\Rightarrow \quad \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2''(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

Thus $c_1 y_1 + c_2 y_2$ is a sol^{n} of the IVP

$$y'' + p(x) y' + q(x) y = 0, \quad \begin{array}{l} y(x_0) = 0 \\ y'(x_0) = 0 \end{array}$$

But the f^{n} $y \equiv 0$ is also a sol^{n} of the same IVP. and so by the uniqueness part of Thm 1, we have

$$c_1 y_1 + c_2 y_2 = 0 \quad \underline{\text{everywhere}} \text{ on } I.$$

Since c_1, c_2 cannot bothe be 0

as $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, this must mean

that y_1, y_2 are lin dep. on I.

Finally, we have just seen in b)

that if $W(y_1, y_2) = 0$ at some $x_0 \in I$,
then y_1, y_2 are lin ind. on I. But
part a) we showed that if y_1, y_2
are lin dep. on I, then $W(y_1, y_2) = 0$
<u>everywhere</u> on I. Result now follows.

$\boxed{/\!/}$

Ex. $y_1 = \cos(\omega x)$, $y_2 = \sin(\omega x)$ are both solns of $y'' + \omega^2 y = 0$. Their Wronskian is

$$W(\cos(\omega x), \sin(\omega x))$$

$$= \begin{vmatrix} \cos(\omega x) & \sin(\omega x) \\ -\omega\sin(\omega x) & \omega\cos(\omega x) \end{vmatrix}$$

$$= \omega\cos^2(\omega x) + \omega\sin^2(\omega x)$$

$$= \omega$$

By Thm 2, y_1, y_2 are lin ind. (on $\mathbb{R}$) iff $\omega \neq 0$.

Can also see this by taking the quotient

$\frac{y_2}{y_1} = \tan\omega x$ which is non-const iff $\omega \neq 0$.

<u>Ex</u>. $y_1 = e^x$, $y_2 = xe^x$ are both solns of $y'' - 2y' + y = 0$. Their Wronskian is

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}$$

$$= (x+1)e^{2x} - e^{2x}$$

$$= e^{2x} \neq 0$$

and so y_1, y_2 are lin ind (on $\mathbb{R}$)

A General Solution Includes all Solutions

Thm 3 Existence of a general solⁿ.

If $p(x)$ and $q(x)$ are cts on an open interval I, then

$$y'' + p(x)y' + q(x)y = 0$$

has a general solⁿ on I.

Pf. By thm 1, the ODE has a solⁿ $y_1(x)$ on I satisfying the initial conditions

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0$$

and another solⁿ $y_2(x)$ on I satisfying

$$y_2(x_0) = 0, \quad y_2'(x_0) = 1$$

(Here x_0 is just any arbitrarily chosen pt of I).

Then at x_0

$$W(y_1, y_2) = \det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 1 \neq 0.$$

Thus by Thm 2, y_1, y_2 are lin ind on I and so by the defn of a general soln,

$$y = c_1 y_1 + c_2 y_2$$

where c_1, c_2 are arb. consts, is a general soln on I.

We finally show that a general solution is as general as it can possibly be.

<u>Theorem 4</u> A General Solution Includes all Solutions

If the ODE $y'' + p(x)y' + q(x)y = 0$ has cts. coeffts $p(x), q(x)$ on an open interval I, then every sol^{n} $y = Y(x)$ of this ODE is of the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

where y_1, y_2 is any basis of sol^{ns} of the ODE and C_1, C_2 are suitable consts.

Pf. Let Y be any sol^{n} of the ODE, let y_1, y_2 be a basis of sol^{ns} and let x_0 be any pt of I.

Since y_1, y_2 is a basis, $W(y_1, y_2)$ is non-zero everywhere on I and in particular at x_0.

Thus the matrix

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible and so we can find C_1, C_2 (unique) s.t.

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} Y(x_0) \\ Y'(x_0) \end{bmatrix} . \quad (*)$$

If we now let $y^* = C_1 y_1 + C_2 y_2$, then y^* is a soln of the ODE on I by Thm 1 in § 2.1 (superposition of solns for homogr linear 2nd order ODEs).

Also by ⊛

$$y^*(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) = Y(x_0)$$

$$(y^*)'(x_0) = C_1 y_1'(x_0) + C_2 y_2'(x_0) = Y'(x_0)$$

ie y^*, Y have the same initial conditions at x_0 and by the uniqueness part of Thm 1, we must have that

$$y^* = Y \quad \text{everywhere on } I$$

Thus, given any sol$^{\underline{n}}$ Y of the ODE, we can find C_1, C_2 s.t.

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

everywhere on I as required. ▨