Chapter 2 Second-Order Linear ODEs

§2.1 Homogeneous Linear ODEs of Second Order

A second-order ODE is called linear if it can be written

$$(*) \qquad y'' + p(x)y' + q(x)y = r(x) .$$

Otherwise we say the second-order ODE is nonlinear.

Note that such an eqn is linear in y, y', y''. If the eqn begins with $f(x)\, y''$ instead of just y'', we can obtain the standard form $(*)$ by dividing by $f(x)$.

$$f(x)y'' + p(x)y' + q(x)y = r(x)$$

$$\downarrow$$

$$y'' + \frac{p(x)}{f(x)}y' + \frac{q(x)}{f(x)}y = \frac{r(x)}{f(x)}$$

<u>Note</u>. We will need to be careful here about places where $f(x) = 0$.

If $r(x) \equiv 0$, ⊛ reduces to

$$y'' + p(x)y' + q(x)y = 0$$

which is a <u>homogeneous</u> linear ODE of second order. Otherwise if $r(x) \not\equiv 0$, we say the ODE is <u>inhomogeneous</u>.

Eg.

$$y'' + 25y = e^{-x}\cos x$$

is inhomogeneous.

$$xy'' + y' + xy = 0$$

is homogeneous and we can write it in standard form as

$$y'' + \frac{1}{x}y' + y = 0.$$

$$y''y + (y')^2 = 0$$

is nonlinear.

The fns p, q in

$$y'' + p(x)y' + q(x)y = 0$$

are called the <u>coefficients</u> of the ODE.

A fn $h(x)$ defined on an open interval I is a <u>solution</u> of the second-order ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

if h is twice differentiable and

$$h''(x) + p(x)h'(x) + q(x)h(x) = r(x)$$

for each x in I.

Homogeneous Linear ODEs : Superposition Principle

The solns of a homogeneous second-order linear ODE have the following very important property.

Thm 1 Fundamental Theorem for Homogeneous Second-Order Linear ODEs.

If y_1 and y_2 are both solutions on an open interval I of

$$y'' + p(x)y' + q(x)y = 0,$$

then so is any linear combination $c_1 y_1 + c_2 y_2$. In particular the sum $y_1 + y_2$ is a soln as is $c y_1$ for any constant c.

Pf. Let y_1, y_2 be solns on I and set $y = c_1 y_1 + c_2 y_2$.

Then $y'' + p(x)y' + q(x)y$

$$= (c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2)$$

$$= (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2)$$

$$= c_1 (y_1'' + p(x) y_1' + q(x) y_1) + c_2 (y_2'' + p(x) y_2' + q(x) y_2)$$

$= c_1(0) + c_2(0)$ as y_1, y_2 are solns

$= 0$

Thus y is also a soln of the ODE. $\boxed{}$

<u>CAUTION</u> This holds only for <u>homogeneous</u> ODEs.

<u>Ex:</u> One can check easily that $\cos x$ & $\sin x$ are both sol^{ns} of

$$y'' + y = 0$$

Thus by thm 1, $c_1 \cos x + c_2 \sin x$ is also a sol^n for any consts c_1, c_2.

One can also easily check that $\cos x + 1$, $\sin x + 1$ are both sol^{ns} of

$$y'' + y = 1.$$

However $\cos x + \sin x + 2$ is not a sol^n (note that this eq^n is inhomog.).

Ex. One can check easily that

$y = x^2$ and $y = 1$ are both sol$^{\underline{ns}}$

of the nonlinear ODE

$$y''y - xy' = 0$$

However, their sum $1 + x^2$ is not a sol$^{\underline{n}}$. Neither is $-x^2$!

Thm 1 shows that the sol$^{\underline{ns}}$ of a second order homog. ODE form a vector space (whose vectors are actually functions). We can also talk about linear independence and basis in this context.

Two functions y_1, y_2 defined on an open interval I are called <u>linearly independent</u> on I if

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \text{everywhere on } I$$

implies $c_1 = 0,\ c_2 = 0$

Otherwise, y_1, y_2 are <u>linearly</u> <u>dependent</u>.

In this case, $\exists\ c_1, c_2$ not both 0

s.t.

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \text{everywhere on } I.$$

Note that in this case either

$$y_2 = -\frac{c_1}{c_2} y_1 \quad (\text{if } c_2 \neq 0)$$

or

$$y_1 = -\frac{c_2}{c_1} y_2 \quad (\text{if } c_1 \neq 0)$$

(or both).

In either case we see that if y_1 and y_2 are linearly dependent, then they are <u>proportional</u>.

Thus two fns y_1, y_2 are lin ind. iff they are not proportional.

<u>Defn</u>. A pair of solns of

$$y'' + p(x)y' + q(x)y = 0$$

which is lin. ind. on an open interval I is called a basis of solns of the ODE on I.

If y_1, y_2 are a basis of solns of the ODE on I, then

$$y = c_1 y_1 + c_2 y_2$$

where c_1, c_2 are arb. constants is called the <u>general solution</u> of the ODE on I. If c_1, c_2 are fixed, we have a <u>particular solution</u> on I.

In order to fix values of the consts c_1, c_2 as in the first order case we need initial conditions.

$$y(x_0) = K_0, \quad y'(x_0) = K_1.$$

Here we have two constants to determine, so we need two initial conditions.

A problem of the form

$$y'' + p(x)y + q(x) = r(x)$$

$$y(x_0) = K_0, \quad y'(x_0) = K_1$$

is called an <u>initial value problem</u> (IVP).

Ex. Solve the IVP

$$y'' + y = 0, \quad y(0) = 3, \quad y'(0) = \frac{1}{2}.$$

(on all of $\mathbb{R}$).

Sol^n.

Step 1 General Sol^n.

We already saw $y_1 = \cos x$, $y_2 = \sin x$ are sol^{n}s of the ODE. Also they are not proportion since otherwise the ratio

$\frac{y_2}{y_1} = \tan x$ would be a constant fn

which is clearly not the case. Thus y_1, y_2 are lin. ind. and the general sol^n is

$$y = c_1 \cos x + c_2 \sin x$$

Step 2 Particular Sol^n

Need $y' = -c_1 \sin x + c_2 \cos x$.

Then $y(0) = 3 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = 3$

$y'(0) = \frac{1}{2} \Rightarrow c_1 \cdot 0 + c_2 \cdot 1 = \frac{1}{2}$.

ie $$c_1 = 3$$
$$c_2 = \frac{1}{2}$$

and so the particular soln of our IVP is

$$y = 3\cos x + \frac{1}{2}\sin x.$$

Ex. Solve the IVP

$$y'' - y = 0, \quad y(0) = 6, \quad y'(0) = -2.$$
(on all of $\mathbb{R}$).

Step 1. Gen soln.

One checks easily that $y_1 = e^x$, $y_2 = e^{-x}$ are solns of the ODE.

The ratio $\frac{y_2}{y_1} = \frac{e^{-x}}{e^x} = e^{-2x}$ is a non-const fn and so y_1 and y_2 are lin ind and the general soln is

$$y = c_1 e^x + c_2 e^{-x}.$$

Step 2 Particular Sol^n

Again need

$$y' = c_1 e^x - c_2 e^{-x}.$$

$y(0) = 6 \Rightarrow c_1 e^0 + c_2 e^{-0} = 6$

$y'(0) = -2 \Rightarrow c_1 e^0 - c_2 e^{-0} = -2$

Get

$$c_1 + c_2 = 6$$
$$c_1 - c_2 = -2$$

Add

$$2c_1 = 4$$

$$c_1 = 2$$

$$c_2 = 6 - c_1$$
$$= 6 - 2 = 4$$

Thus
$y = 2e^x + 4e^{-x}$ is the required sol^n of the IVP.

Why linear independence is so important.

Suppose in the last example instead of taking $y_2 = e^{-x}$, we took $y_2 = ke^x$, a const. multiple of y_1.

Our 'general' sol^n would then be

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 e^x + c_2 k e^x$$

with $y' = c_1 e^x + c_2 k e^x$

$$y(0) = 6 \Rightarrow c_1 + c_2 k = 6$$

$$y'(0) = -2 \Rightarrow c_1 + c_2 k = -2$$

which is impossible.

This is not just an isolated pathology. In general, if y_1, y_2 are not lin ind., we can only hope to satisfy at most one of the initial conditions.

Finding a Basis if One Solution is Known: Reduction of Order

<u>Idea</u> If we already know one solution of a second order ODE, we can find a second lin ind. solⁿ by solving a first order ODE (hence the name).

<u>Ex</u>. Find a basis of solⁿˢ of

$$(x^2-x)y''-xy'+y=0$$

It is easy to see that $y=x$ is a solⁿ (here $y'=1$, $y''=0$).

Idea is to let
$$y = uy_1 = ux$$
$$y' = u'x + u$$
$$y'' = u''x + 2u'$$

and subst into the ODE to get

$$(x^2-x)(u''x+2u') - x(u'x+\cancel{u}) + \cancel{ux} = 0$$

$-ux$ & ux cancel and we get

$$(x^2-x)(u''x+2u') - x^2u' = 0$$

$$(x^3-x)u'' + (2x^2-2x-x^2)u' = 0$$

Divide by x

$$(x^2-x)u'' + (x-2)u' = 0$$

If we now let $v = u'$, then $v' = u''$ and

$$(x^2-x)v' + (x-2)v = 0$$

which is <u>first order</u> in v.

Separation of variables gives

$$\frac{dv}{v} = -\frac{x-2}{x^2-x}dx = \left(\frac{1}{x-1} - \frac{2}{x}\right)dx$$

(partial fractions).

Integrate

$$\ln|v| = \ln|x-1| - 2\ln|x| = \ln\left|\frac{x-1}{x^2}\right|$$

n.b. don't need a const of integration as we want just one soln.

Taking exponents.

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$$

$$u = \int v\,dx = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \ln|x| + \frac{1}{x}$$

(again ignore the const of integration).

So $y_2 = ux = x\ln|x| + 1$.

and since $y_1 = x$, $y_2 = x\ln|x| + 1$ are lin ind (their quotient is not a const), we have a basis of solns (valid for $x \neq 0$)

In general, for a second-order linear homog. ODE

$$y'' + p(x)y' + q(x)y = 0$$

in standard form (which we need for our method to work),

let y_1 be a known sol^{n} on an open interval I and we want to find a basis (i.e. a second lin ind. sol^{n} y_2 on I).

To get y_2, subst.

$$y = y_2 = uy_1, \quad y' = y_2' = u'y_1 + uy_1'$$

$$y'' = y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

into the ODE.

This gives

$$u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0$$

Collecting terms in u'', u', u gives

$$u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0$$

<u>Important Point</u> Since y_1 is a soln.,

$y_1'' + py_1' + qy_1 = 0$, so the part involving u disappears and we are left with

$$u''y_1 + u'(2y_1' + py_1) = 0$$

Divide by y_1

$$u'' + u'\frac{(2y_1' + py_1)}{y_1} = 0$$

$$u'' + u'\left(\frac{2y_1'}{y_1} + p\right) = 0$$

This eqn involves only u', u'' and if we let $u' = U$, so that $u'' = U'$, then

$$U' + U\left(\frac{2y_1'}{y_1} + p\right) = 0$$

This is the desired (reduced) first order ODE.

Separation of variables gives

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right)dx$$

and if we then integrate

$$\ln|U| = -2\ln|y_1| - \int p\,dx\,.$$

Taking exponents, we have

$$U = e^{-2\ln|y_1| - \int p\,dx}.$$

$$\boxed{U = \frac{1}{y_1^2} e^{-\int p\,dx}}$$

Now $U = u'$ and $y_2 = y_1 u$, so

$$\boxed{y_2 = y_1 \int U\,dx}$$

Note that the quotient $\frac{y_2}{y_1} = \int U\,dx$ cannot be const (since $U > 0$), so that y_1, y_2 are lin. ind. and form a basis of sol$^{\text{ns}}$.